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Experiment of boundedly rational route choice behavior and the model under satisficing rule ☆

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ABSTRACT

In this paper, we study the boundedly rational route choice behavior under the Simon's satisficing rule. A laboratory experiment was carried out to verify the participants' boundedly rational route choice behavior. By introducing the concept of aspiration level which is specific to each person, we develop a novel model of the problem in a parallel-link network and investigate the properties of the boundedly rational user equilibrium (BRUE) state. Conditions for ensuring the existence and uniqueness of the BRUE solution are derived. A solution method is proposed to find the unique BRUE state. Extensions to general networks are conducted. Numerical examples are presented to demonstrate the theoretical analyses.

1. Introduction

Route choice problem refers to the decision-making of route (or path) use between origins and destinations in transportation networks. The question of interest is how travelers are distributed among all possible routes. To solve this, a rule by which travelers choose routes should be known in advance by the modeler. By assuming all road users behave in a completely rational way and seek to minimize their own disutility, Wardrop (1952) defined a state resulted from route choices of many individuals, so-called user equilibrium (UE). At the UE state, no user can further improve her or his utility by unilaterally changing routes. By relaxing some of the behavioral restrictions implied in a strictly deterministic disutility minimization rule, Daganzo and Sheffi (1977) developed a stochastic user equilibrium (SUE) model that considers the travelers' imperfect perceptions of travel times. In this model, the link travel time perception error is treated as a random variable which follows some known probability distribution. The Gumbel and normal distributions are two commonly used ones, which result in the well-known logit-based and probit-based route choice models, respectively (Dial, 1971; Daganzo and Sheffi, 1977). The SUE is achieved when users can no longer change their perceived utility. Existence and uniqueness of UE or SUE in general networks have been well investigated in literature, including the solution methods for obtaining these two states, see Sheffi (1985), Yang and Huang (2005) and Prato (2009) for more details. Recently, Kitthamkesorn and Chen (2013) proposed a path size Weibit stochastic user equilibrium model which adopts the Weibull distributed random error term to handle the route-specific perception variance. Another decision rule for route choice is based on the regret theory (Bekhor et al., 2012; Chorus, 2012a, 2012b).

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As a relaxation of the perfect rational and optimization assumption, Mahmassani and Chang (1987) introduced the concept of indifference band and analyzed the boundedly rational user equilibrium (BRUE) in the standard single-link bottleneck network. They studied the existence, uniqueness, and stability properties of the BRUE. The width of the band was calibrated by empirical observation (Mahmassani and Jayakrishnan, 1991; Hu and Mahmassani, 1997; Mahmassani and Liu, 1999).

Based on the concept of indifference band, Lou et al. (2010) defined a concept of the acceptable path and systematically examined the mathematical properties of BRUE in the static traffic assignment context. They proved that the set of BRUE flow distributions is generally non-convex and non-empty as pointed in Mahmassani and Chang (1987). One of the difficulties in carrying out the BR traffic assignment is that the BRUE state is generally not unique. Wendin et al. (2010) formulated the problem of finding a BRUE flow distribution as a mathematical program with complementarity constraints and searched for a strongly stationary solution using the manifold sub-optimization. Di et al. (2013) proposed a methodology of generating various path combinations and constructed the set of all BRUE solutions in traffic networks with fixed demand.

In the above studies about boundedly rational travel behavior, a key concept is the indifference band (Mahmassani and Chang, 1987) that commuters are assumed to behave as if they had an "indifference band" of tolerable schedule delay and the BRUE state of the system is reached when all users are satisfied with their current choices, and thus do not intend to switch again. Mahmassani and Chang (1987) also noted that while the boundedly rational perspective is not standard in transportation demand literature, it is quite widely accepted in a variety of disciplines as a plausible model of individual decision making, particularly when facing complex decision situations under limited information. For example, various satisficing models are well established in marketing research and consumer behavior, which is of particular relevance in light of analogy between commuting behavior and consumer repurchase decisions.

In this paper, we investigate the boundedly rational route choice behavior that reflects the realistic decision making in route choice problem, aiming at understanding this kind of behavior deeply. Many studies showed that in reality users do not always choose the shortest paths, and the classical Wardrop user equilibrium assignment model cannot give accurate prediction of traffic flow patterns (Nakayama et al., 2001; Avineri and Prashker, 2004; Morikawa et al., 2005). The theory of bounded rationality may provide better prediction to actual traffic flow pattern than the traditional behavior economic theory (Camerer and Fehr, 2006). The first to address the notion of bounded rationality was Simon (1955). Simon suggested a theory of bounded rationality based on satisficing and aspiration levels due to the informational and computational limits of human rationality. The basic idea of the Simon's approach is that people set up the aspiration levels on a goal variable and search for alternatives that can guarantee them. In the simplest case, the search process continues until a satisfactory alternative is found. Simon (1959) coined the term "satisficing" to describe this approach. Satisficing acts as a "stop rule". The concept of bounded rationality has been extensively studied in the economic and psychology literature. It has been shown that bounded rationality is important in many contexts (see de Palma et al. (1994), Conlisk (1996) and references cited therein). In the route choice context, real-world users often choose the first available route for ensuring certain aspiration level being reached. The explicit description of satisficing rule will be given in Section 3.

We first carry out a laboratory experiment to examine the participants' bounded rationality in making route choice decisions. Similar experiments on route choice behavior have already done by other scholars (Selten et al., 2007; Hartman, 2012). Selten et al. (2007) finished the laboratory experiments of a day-by-day route choice game with two parallel roads, and reported that aggregate road choices are accounted for quite well by the Nash equilibrium predictions and large fluctuations do not diminish with individuals' experiences. Hartman (2012) investigated how people respond to the use of a road toll, and found that the toll comes very close to achieving an efficient use of the traffic network. We will collect the users' perceived travel time costs on two routes and examine whether the users' decisions are rational.

We then make a theoretical investigation of boundedly rational route choice behavior under satisficing rule through introducing the concept of aspiration level. To our best knowledge, though there have been several studies examining the aspects of "attribute threshold" (Swait, 2001; Cantillo and de Dios Ortúzar, 2005; Cantillo et al., 2006) and "information processing strategy" (Chorus et al., 2006; Zhu and Timmermans, 2010) in boundedly rational behavior, there is not yet any research targeting the concept of aspiration level in route choice behavior when considering congestion effect. The concept of aspiration level is different from the concept of indifference band in the following two aspects. (i) Their decision mechanisms are different. In reality, a user may consider the difference between the route travel time and his/her aspiration level when choosing a route, not the indifference band that is the travel time difference between his/her current route and the best route. (ii) The indifference band is not the true sense of the bounded rationality proposed by Simon (1955) in behavioral perspective, while the aspiration level can reflect bounded rationality better. In this paper, we assume that the aspiration levels of all individuals at BRUE state are fixed. Actually, aspiration levels are not permanently fixed, but time-varying. Our future research will study the adjustment of aspiration levels. It should be noted that a similar concept, called acceptance level, is widely used in the transportation associated literature, in this paper, however, we prefer to use the term aspiration level for highlighting Simon's original work.

The rest of the paper is organized as follows. In the next section, we present the results of a laboratory experiment dealing with route choice behavior. In Section 3, we give some definitions and assumptions to be used throughout the whole paper. Section 4 analyses the properties of BRUE state under satisficing rule together with a numerical illustration. Conditions for ensuring the existence and uniqueness of the BRUE solution are derived and a solution method is proposed in this section. Extensions to general networks are discussed in Section 5. Section 6 concludes the paper.

2. Experimental study on route choice

2.1. Experimental design

All participants who were recruited from Beihang University were required to choose the main road M or the side road S for traveling from A to B (see Fig. 1). The experiment was conducted in a computerized laboratory with multiple terminals in May 1, 2014. The game situation is identical for every participant. The number of participants was 18. All of them are graduate students, 10 female and 8 male.

All participants were told that the travel time of a road will increase with the traffic flow on that road due to congestion effect. If traffic is the same on both roads, the main road's travel time is less than the side road. The travel time costs of these two roads are computed by two linear functions $t_M = 6 + 2x_M$ and $t_S = 12 + 3x_S$, respectively. The flows on these two roads are denoted by x_M and x_S.

All participants, in each of the total 30 runs, were informed to input their perceived travel time costs of these two roads, their aspiration levels (or acceptance level), and finally make the road choice decisions. At the beginning of each run, participants can receive feedbacks associated with the previous run. The feedbacks which may help their decisions include the following information: (i) the real travel costs of two roads in the preceding run, (ii) their own road choice decisions in the preceding run, (iii) their own inputted travel costs in the preceding run, (iv) their own accumulative payoffs, and (v) the number of the current run.

At the beginning of the experiment, each participant holds 1000 RMB (Chinese currency). The total payoff of a participant is $1000 - \sum_{e=1}^{30} t^e$ with $t^e = t_M^e$ if M was chosen and $t^e = t_S^e$ if S was chosen. Additionally, each participant received a show-up fee of 20 RMB.

2.2. Results and discussions

2.2.1. Data analysis

Table 1 lists the road choices of 18 participants in 30 runs. For convenience, in this table we use the numbers 1 and 2 to represent roads M and S, respectively. Fig. 2 shows the number of participants choosing roads M and S respectively. Fig. 3 gives the travel time costs of the two roads.

Let x_i^j denote the number of participants who choose and use road i in run j, $i = (M, S)$. Let t_i^j be the travel time cost of road i in run j. Now we analyze the road choice decisions of users in the first two runs. From Figs. 2 and 3, we know that $x_M^1 = 15$, $x_S^1 = 3$, $t_M^1 = 36$, $t_S^1 = 21$; $x_M^2 = 11$, $x_S^2 = 7$, $t_M^2 = 28$, $t_S^2 = 33$. From Table 1, we find that Users 1, 10 and 15 made irrational choices. These three users indeed chose a lower cost road (i.e., Road S) in Run 1, but turned to choose the higher cost road (i.e., Road M after Run 1) in Run 2. Other eight users (i.e. Users 3, 8, 9, 13, 14, 16, 17 and 18) also made irrational choice decisions, they chose the higher cost road (i.e., Road M) in Run 1, but still chose this higher cost road (i.e., Road M) in Run 2. The rest of users seem to be rational, i.e., they changed their choices by shifting from higher cost road to lower cost road.

By analyzing the data of total 30 runs, we can observe that (1) there is no convergence to the theoretical equilibrium of the two roads' costs. Fluctuations exist until the end of the experiment. It is consistent with the findings by Selten et al. (2007) in which 200 runs were conducted; (2) irrational choice decisions always exist.

The theoretical equilibrium requires that 12 users choose Road M and 6 users Road S, generating an equilibrium travel cost of 30 on both roads. Clearly, the classical equilibrium analysis cannot predict the fluctuations observed in our experiment. In addition, by comparing the choice probability and standard deviation, we find that the symmetric mixed equilibrium (this concept is well known as stochastic user equilibrium in transportation science) grossly overestimates the size of fluctuations. In other words, the stochastic user equilibrium analysis also cannot accurately predict the fluctuations. This is again consistent with the findings by Selten et al. (2007).

For having a better explanation to the persisting fluctuations, we may have to seek theoretical supports from boundedly rational route choice behavior. According to the viewpoint by Simon (1955), users are boundedly rational and make decisions in terms of the so-called satisficing rule. If users are satisfied with their current decisions, they will not change more. In other words, users will not change their road choice if the current road cost is not larger than his aspiration level, and vice versa. This is the reason why we conduct the research presented in this paper.

In the next subsection we give some explanations to these two findings using the theory of boundedly rational route choice behavior.

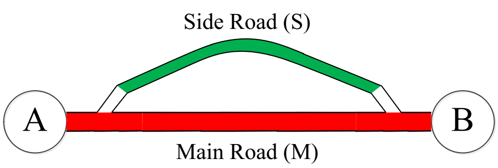

Fig. 1. An experimental network having a side road (S) and a main road (M).

Table 1
The choices of 18 participants in 30 runs (1 and 2 represent *M* and *S* respectively).

Runs	Participants																	
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	2	1	1	1	1	1	1	1	1	1	2	1	1	1	2	1	1	1
2	1	2	1	2	2	2	2	1	1	2	1	2	1	1	1	1	1	1
3	1	1	1	2	1	1	1	2	1	2	1	1	1	1	1	1	1	1
4	1	1	1	1	1	2	1	2	1	1	2	1	1	1	1	2	2	1
5	2	2	1	2	1	2	1	2	1	1	2	2	1	1	2	2	2	2
6	1	1	1	2	2	1	2	1	1	1	1	2	1	1	2	1	1	1
7	1	2	1	2	1	2	1	1	1	1	1	1	1	1	1	2	1	2
8	2	2	1	1	1	2	1	2	1	2	2	1	1	1	1	1	2	2
9	1	1	1	2	2	1	2	1	1	2	1	2	1	1	2	1	1	1
10	1	2	1	2	2	2	2	1	1	2	2	1	1	1	2	1	2	1
11	1	1	1	1	2	2	1	1	1	2	1	2	1	1	2	1	1	1
12	1	2	1	2	1	2	1	2	1	2	2	2	1	1	1	2	2	1
13	1	1	1	1	2	2	2	1	1	2	1	1	1	1	2	1	1	2
14	1	1	1	2	1	2	1	1	1	2	1	1	1	1	1	1	1	2
15	2	2	1	2	1	2	1	2	1	2	1	1	1	1	1	2	2	2
16	2	2	1	1	2	1	2	1	1	2	2	1	1	1	2	1	1	2
17	2	2	1	2	2	1	2	1	1	2	1	1	1	1	2	1	1	1
18	2	2	1	1	2	2	2	2	1	2	2	1	1	1	1	1	1	1
19	1	1	1	1	2	1	1	1	1	2	1	1	1	1	2	1	1	1
20	2	2	1	2	2	2	1	1	1	1	2	1	1	2	2	2	2	1
21	1	1	1	1	2	1	1	1	1	1	1	2	1	1	2	1	1	1
22	1	2	1	1	1	1	1	1	1	2	2	1	1	1	2	1	2	1
23	1	1	1	1	1	2	1	1	1	1	1	1	1	1	1	1	1	1
24	1	1	1	1	1	1	1	2	1	2	1	1	1	2	1	2	1	2
25	1	1	1	1	2	1	1	2	1	2	1	1	2	1	2	1	1	2
26	2	1	1	1	1	1	1	2	1	2	2	1	1	1	2	1	1	1
27	2	1	1	1	1	2	1	2	1	2	2	1	1	1	2	2	2	1
28	1	2	1	1	2	1	2	1	2	1	1	1	1	1	1	1	1	1
29	1	1	1	2	1	1	2	1	2	1	2	1	1	1	2	1	2	1
30	2	2	1	2	1	2	2	1	1	2	1	1	1	1	2	1	1	1

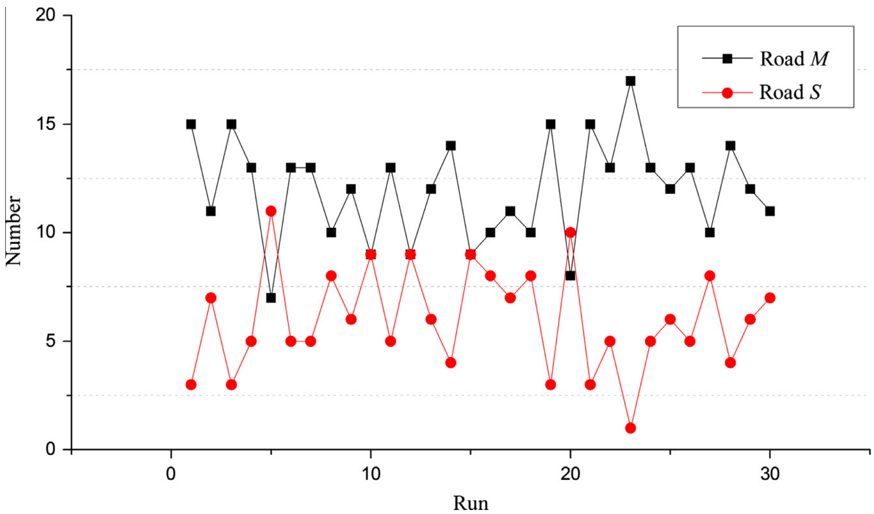

Fig. 2. The number of participants choosing *M* and *S*.

2.2.2. Relationship between perceived travel costs and aspiration levels

In the above Section 2.2.1, we find that irrational choice decisions always exist in total 30 runs. Before users make the road choice decisions in the new run, they do not know the exact travel costs of two roads due to the informational and computational limits of human rationality. They make decision based on the perceived road travel costs and their own aspiration levels. Hence, their choices look like irrational.

Additionally, the persisting fluctuations may be explained by the dynamic adjustment process of perceived road travel costs and aspiration levels. This is one of topics to be further investigated by developing a learning model.

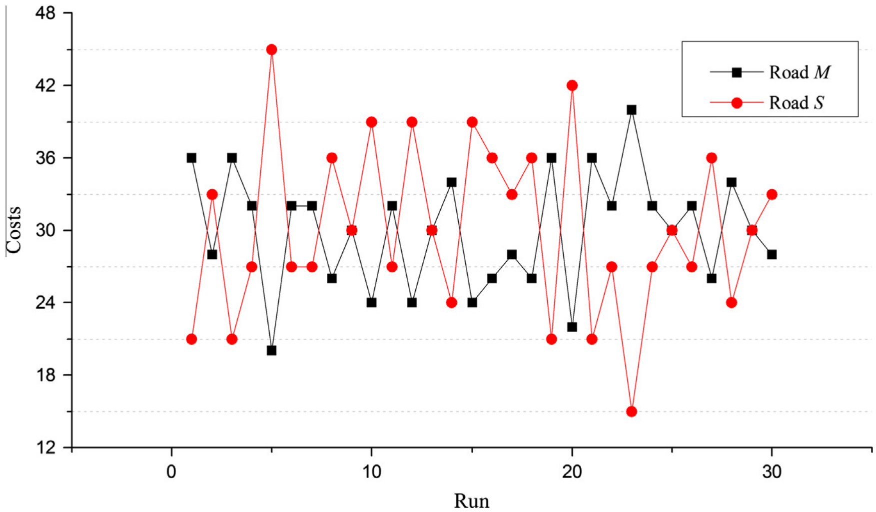

Fig. 3. The travel time costs of the two roads.

Table 2
Proportion of choosing Road *M* in 30 runs.

User ID	Proportion of choosing *M*	User ID	Proportion of choosing *M*
10	0.33	7	0.63
15	0.40	17	0.67
6	0.47	18	0.70
2	0.53	16	0.73
4	0.53	12	0.77
5	0.53	9	0.93
11	0.57	14	0.93
8	0.63	13	0.97
1	0.63	3	1.00

Table 3
Perceived travel costs (PC) and aspiration levels (AL) of User 6 in 30 runs.

Run	PC of *M*	PC of *S*	AL	Choice	Run	PC of *M*	PC of *S*	AL	Choice
1	30	35	35	1	16	36	39	36	1
2	25	20	24	2	17	29	31	30	1
3	33	35	35	1	18	36	26	30	2
4	25	22	24	2	19	28	35	30	1
5	25	20	22	2	20	35	22	25	2
6	20	30	25	1	21	20	40	25	1
7	30	28	28	2	22	29	33	30	1
8	35	28	28	2	23	33	25	30	2
9	26	30	28	1	24	25	33	30	1
10	28	27	27	2	25	28	32	28	1
11	33	28	28	2	26	26	30	28	1
12	35	28	30	2	27	41	29	32	2
13	36	24	30	2	28	29	39	29	1
14	38	29	29	2	29	22	33	25	1
15	29	30	30	2	30	33	22	25	2

In our experiment, we collected the data of some users' perceived road travel costs and aspiration levels (or acceptance level) before each run for verifying the boundedly rational road choice behavior.

We first give the proportion of choosing Road *M* for each user in 30 runs, as shown in Table 2. We find that there are 15 users (83% of all users) whose proportions of choosing *M* in 30 runs are higher than 53%. This means that most users prefer road *M* to road *S*. We choose three representative users, Users 6, 18 and 9, to further analysis their road choice behavior. Tables 3–5 show the collected data of these three users. From these three tables, we can find that (1) the value of aspiration level is a little higher than the perceived travel cost of a road that users plan to choose in each run; (2) the perceived travel cost of a road that users plan to choose is lower than that of another road in each run; (3) the values of perceived travel costs and aspiration levels are time-varying. These findings give an explanation to the persisting fluctuations.

Table 4
Perceived travel costs (PC) and aspiration levels (AL) of User 18 in 30 runs.

Run	PC of *M*	PC of *S*	AL	Choice	Run	PC of *M*	PC of *S*	AL	Choice
1	30	40	35	1	16	32	24	25	2
2	20	48	25	1	17	24	42	24	1
3	22	48	25	1	18	24	42	24	1
4	28	40	30	1	19	26	36	30	1
5	36	20	25	2	20	20	37	20	1
6	20	45	30	1	21	24	42	25	1
7	30	24	25	2	22	24	42	25	1
8	30	22	25	2	23	22	44	25	1
9	22	42	25	1	24	32	28	30	2
10	24	42	25	1	25	32	28	30	2
11	24	42	24	1	26	28	40	28	1
12	24	40	24	1	27	24	42	28	1
13	32	24	24	2	28	24	42	25	1
14	32	28	30	2	29	24	34	25	1
15	32	24	24	2	30	24	42	25	1

Table 5
Perceived travel costs (PC) and aspiration levels (AL) of User 9 in 30 runs.

Run	PC of *M*	PC of *S*	AL	Choice	Run	PC of *M*	PC of *S*	AL	Choice
1	30	36	35	1	16	20	33	25	1
2	36	30	40	1	17	20	34	25	1
3	29	31	30	1	18	23	35	25	1
4	24	27	25	1	19	21	32	25	1
5	33	36	35	1	20	23	33	25	1
6	29	30	30	1	21	29	39	30	1
7	33	30	35	1	22	23	31	25	1
8	30	24	30	1	23	36	24	36	1
9	27	33	30	1	24	36	37	36	1
10	19	29	25	1	25	31	21	35	1
11	20	36	25	1	26	36	26	36	1
12	27	37	30	1	27	27	37	30	1
13	34	30	35	1	28	30	27	27	2
14	25	35	30	1	29	31	24	25	2
15	26	35	30	1	30	27	37	30	1

In summary, the above analyses verify that users do not behave in a way of purely seeking for utility maximization but in a boundedly rational manner. The relationship between numbers of road choice changes and users' total payoff, as well as road choice decision response modes, and the dynamic adjustment process of aspiration levels and perceived travel costs are our future research topics. In the following sections, we only focus on the static BRUE case with different fixed aspiration levels and conduct a theoretical analysis under such an assumption that users have identical route preference but different satisfied road travel time costs.

3. Definitions and assumptions

Consider a general traffic network $G(N, L)$, where N is the set of nodes and L the set of directed links. Let W be the set of OD pairs and R_w the set of routes connecting OD pair $w \in W$. Routes are assumed to be acyclic. It is assumed that the travel demands in the network are fixed and denoted by a column vector $\mathbf{d} = (d_w, w \in W)^{\mathrm{T}}$, where d_w is the travel demand between OD pair $w \in W$. f_{rw} is the traffic flow on route $r \in R_w$, and the route flow vector is denoted by $\mathbf{f} = (f_{rw}, r \in R_w, w \in W)^{\mathrm{T}}$. x_a is the traffic flow on link $a \in L$, and the link flow vector is denoted by $\mathbf{x} = (x_a, a \in L)^{\mathrm{T}}$. The link-route incidence matrix is denoted as $\mathbf{\Delta} = (\delta_{ar}, a \in L, r \in R_w, w \in W)$, where $\delta_{ar} = 1$ if route r uses link a and 0 otherwise. The following relationships hold,

$$x_a = \sum_{r \in R_w} f_{rw} \delta_{ar}, \quad a \in L, \tag{1}$$

$$d_w = \sum_{r \in R_w} f_{rw}, \quad w \in W, \tag{2}$$

$$f_{rw} \geqslant 0, r \in R_w, \quad w \in W. \tag{3}$$

The feasible set of link flows is given by X = {$\mathbf{x}$: there exists a vector $\mathbf{f}$ such that (1)–(3) hold}. And, the feasible set of route flows is given by Ω = {$\mathbf{f}$: conditions (2) and (3) hold}.

In this paper, the separable link travel time functions, $t_a(x_a)$, are adopted and assumed to be a monotonically increasing, continuously differentiable function of x_a. Let T_{rw} be the travel time of route r, then

$$T_{rw} = \sum_{a \in L} t_a(x_a)\delta_{ar}, \quad r \in R_w, \quad w \in W. \tag{4}$$

In the following, we give the formal definitions of aspiration level, acceptable route, and BRUE state.

Definition 1. The route travel time that a user can accept is defined as her or his aspiration level. Let $J = \{1, 2, \ldots, d\}$ be the set of all users, the aspiration level of user j is denoted by A^j, $j \in J$. Everyone has his own aspiration level.

Definition 2. A route is acceptable for a user if the travel time of this route is not larger than the user's aspiration level.

Definition 3. A flow distribution is the BRUE state if every user has and uses an acceptable route. In other words, BRUE state is achieved when all users are satisfied with their current choices, and thus do not intend to switch.

Before we proceed to investigate some basic properties of BRUE state under satisficing rule, we here give a behavioral assumption according to the route preference order.

Assumption 1. All users of the same OD pair have the same preference order in choosing routes.

For convenience of the theoretical analysis, though Assumption 1 is strict, it reflects the reality to some extent, and will be employed throughout this paper. One case of Assumption 1 is that the routes for the same OD pair are ordered by increasing free flow route travel times. Karakostas et al. (2011) considered such kind of users, i.e. oblivious users, who choose route relying on simple criteria, such as the shortest route as defined by the actual distances on a map, or they follow a route proposed by a GPS system that actually calculates the shortest (distance-wise) path. In other words, these oblivious users prefer to the shortest route with lowest free flow route travel time. One can even argue that such users may be the majority. Actually, everyone in reality has his own route preference order. True, relaxing this assumption would make the model more realistic. We leave this extension for future research. Based on Assumption 1, the satisficing rule in route choice problem can be explicitly described using the following manner.

For each OD pair, users have the same most preferred route when facing to choose one route from the feasible set of routes and predetermine their own aspiration levels. If this holds, the travel time of the most preferred route is not larger than all users' aspiration levels and all users will choose this route and not switch to other routes. Then, the equilibrium state is achieved. If the travel time of the most preferred route is larger than some users' aspiration levels, then, these users will switch to choose the sub-preferred route. This dynamic process terminates until all users are satisfied with their route choices. We call this equilibrium state as boundedly rational user equilibrium (BRUE) under satisficing rule. In the following analysis, we assume the aspiration levels of all individuals at BRUE state are fixed, and investigate the equilibrium properties.

4. Results for parallel-link networks

In this section, we first derive some analytical results in parallel-link networks with one OD pair. Note that in parallel-link networks, the definitions, "route" and "link", are equivalent. For convenience of formulating the BRUE state mathematically, let A_r^j denote the aspiration level of user j who chooses the link r at BRUE state. Note that the symbol does not mean the aspiration level is specific to each link. Then, the definition of the BRUE state can be formulated as follows:

$$t_r \leqslant \overline{A}_r = \min_{j \in J}\{A_r^j\}, \quad \text{for all } r \in R^+, \tag{5}$$

$$\sum_{r \in R} f_r = d, \tag{6}$$

$$f_r \geqslant 0, \quad r \in R, \tag{7}$$

where $\overline{A}_r$ is the minimal aspiration level of users on link r, $R^+ = \{r : f_r > 0, r \in R\}$ is the set of links with positive flows. Eq. (5) states that the travel time of every actually used link is not larger than all users' aspiration levels on this link. Eq. (6) represents the flow conservation. The non-negativity of link flows is required by (7).

4.1. Properties of the BRUE state

Consider an arbitrary user preference order of links, without loss of generality, we label the order by Arabic numerals, $1, 2, \ldots, n$, and denote the set by $R = \{1, 2, \ldots, n\}$. The fundamental assumption, i.e., Assumption 1, can be replaced by the following Assumption 2.

Assumption 2. For all r and v, $r \leqslant v, r, v \in R$, users prefer to choose link r than link v if both links are acceptable.

Lemma 1. *With Assumption 2, at the BRUE state,* $\max_{j \in J}\{A_v^j\} \leqslant \min_{j \in J}\{A_r^j\}$ *hold for all r and v,* $r \leqslant v, r, v \in R^+$.

Proof. Expression $\max_{j \in J}\{A_v^j\} \leqslant \min_{j \in J}\{A_r^j\}$ means that the minimal aspiration level of users on link r is not lower than the maximal aspiration level of users on link v. We herein prove the lemma by the contradiction method. Suppose at BRUE, there exist two users, one with higher aspiration level chooses link v and another with lower aspiration level chooses link r. According to Assumption 2, however, if one user with lower aspiration level chooses link r, the user with higher aspiration level should choose link r too. Therefore, the aspiration levels of these users who choose link v should be no larger than that of those users who choose link r. □

Lemma 2. *With Assumption 2, at the BRUE state, we have* $t_v \leqslant t_r$*, for all r and v,* $r \leqslant v, r, v \in R^+$*, and* $t_v = t_r$ *if all users are perfectly rational.*

Proof. We provide a proof for this lemma by a contradiction. Suppose $t_v > t_r$ and $f_v > 0$, we then get $\overline{A}_v \geqslant t_v > t_r$. This means both links v and r are acceptable for some users, and all of these users choose link v. This contradicts with Assumption 2. Thus, at BRUE, for all r and v, $r \leqslant v, r, v \in R^+$, relations $t_v > t_r$ are impossible. Clearly, when $t_v = t_r$ for all r and $v, r, v \in R^+$, the travel times of all alternative routes are identical. This is the result of a standard UE assignment in which all users make route choices following a perfectly rational rule. This completes the proof. □

Based on the mathematical definitions of BRUE and the above two lemmas, we have the following theorems.

Theorem 1. *A link flow distribution is link preference-based BRUE, if and only if the following conditions are satisfied for all r and v,* $r \leqslant v, r, v \in R^+$,

$$\overline{A}_r \geqslant t_r \geqslant \max_{j \in J}\{A_v^j\}, \tag{8}$$

$$\sum_{r \in R} f_r = d. \tag{9}$$

Proof. *Necessity.* We only need to prove that if a link flow distribution is BRUE, then, expressions $t_r \geqslant \max_{j \in J}\{A_v^j\}$ hold for all r and v, $r \leqslant v, r, v \in R^+$. We provide a proof by a contradiction. Suppose there exists a user who chooses link v and has an aspiration level larger than the travel time of link r. This contradicts with Assumption 2. Therefore, there are no users such that $t_r < \max_{j \in J}\{A_v^j\}$ hold for all r and v, $r \leqslant v, r, v \in R^+$.

Sufficiency. We have to prove $\overline{A}_r \geqslant t_r$ hold for all $r, r \in R^+$. This means the aspiration level of every user who chooses a link is not lower than the travel time of the chosen link. This is certainly true in terms of Definition 3. In other words, if it is implemented that every user utilizes an acceptable link, the link flow distribution is BRUE. □

Theorem 2. *Denote the BRUE link travel time by* $t_r^B, r \in R^+$*, the UE travel time on link r by* t_r^U*, the minimal Arabic numeral in the set* R^+ *by* $\bar{r}$*. Then, at BRUE state,* $\overline{A}_{\bar{r}} \geqslant t_{\bar{r}}^B \geqslant t_{\bar{r}}^U$ *holds.*

Proof. Clearly, we only need to prove that $t_{\bar{r}}^B \geqslant t_{\bar{r}}^U$ holds. We provide a proof for it by a contradiction. Suppose $t_{\bar{r}}^B < t_{\bar{r}}^U$ holds. From Lemma 2, we have $t_v^B \leqslant t_{\bar{r}}^B < t_{\bar{r}}^U$ for all $v, v \in R^+$. Let f_r^B and f_r^U denote the BRUE flow and UE flow on link r, respectively. In the case that $f_{\bar{r}}^U = 0$, then, $f_{\bar{r}}^B < 0$. This is clearly impossible. In the case of $f_{\bar{r}}^U > 0$, i.e., $t_{\bar{r}}^U$ is the minimal travel time under UE condition. So, $f_v^B < f_v^U$ hold for all $v, v \in R^+$. We then have $\sum_{v \in R^+} f_v^B < \sum_{v \in R^+} f_v^U < d$. This contradicts the flow conservation constraint (6). This completes the proof. □

Corollary 1. *At the BRUE state, we have* $f_{\bar{r}}^B \geqslant f_{\bar{r}}^U$.

Proof. From Theorem 2 and the condition that link travel time function is separable, monotonically increasing and continuously differentiable with respect to link flow, we can easily obtain $f_{\bar{r}}^B \geqslant f_{\bar{r}}^U$. □

Theorem 3. *Let* t^U *denote the minimal link travel time under UE condition. When* $\overline{A}_{\bar{r}} = t^U$*, the BRUE solution reduces to the standard UE solution.*

Proof. Clearly, we only need to prove that when $\bar{A}_{\bar{r}} = t^U$, then $\bar{A}_v = t^U$ hold for all $v, v \in R^+$. From Theorems 1 and 2, we have $t^U = \bar{A}_{\bar{r}} \geqslant t^B_{\bar{r}} \geqslant \bar{A}_v$ and $t^B_{\bar{r}} \geqslant t^U$, then $t^B_{\bar{r}} = t^U \geqslant \bar{A}_v$. If $\bar{A}_v \leqslant t^U$ hold for all $v, v \in R^+$, then $f^B_v < f^U_v$, $\sum_{v \in R^+} f^B_v < \sum_{v \in R^+} f^U_v < d$. This contradicts the link flow conservation constraint (6). Therefore, it holds $\bar{A}_v = t^U$ for all $v, v \in R^+$. This completes the proof. □

Example 1. We employ a simple two-link network to illustrate the above properties. Let f_p and f_q be the link flows, $t_p = 2f_p + 2$ and $t_q = 3f_q + 3$ the link travel times of these two links, respectively. The total demand is $d = 10$. We test two user preference orders, namely **1A** and **1B**, respectively.

1A. Label the links p and q with Arabic numerals 1 and 2, respectively. This means that all users prefer link p to link q. Table 6 gives the link travel time, total travel times, link minimal aspiration level, and equilibrium types subject to different link flow patterns. For the purpose of comparison, the UE and SO flow patterns are also provided. Clearly, the results shown in this table coincide with the foregoing findings exactly.

It should be mentioned that all flow patterns with $f_p < 6.2$ are not at the BRUE state since there does not exist such aspiration level that satisfies the BRUE definition. Consider a flow pattern (6, 4) as an example for illustration. The corresponding link travel time is (14, 15). If it is a BRUE flow pattern, the aspiration level of users choosing link q must not be lower than 15. But, the travel time of link p is 14, which is less than the aspiration level of users choosing link q. According to Assumption 2, some users should shift to link p. Thus, (6, 4) is not at the BRUE state.

Given an aspiration level 16 for some users and another aspiration level 12 for other users, solving the BRUE traffic assignment problem gives a flow pattern (7, 3). This says, 7 users with aspiration level 16 choose link p and 3 users with aspiration level 12 choose link q.

Table 6 also shows that with this preference order, the SO flow pattern generates the minimal total travel time, and the total travel times by various BRUE flow patterns are always larger than that by UE flow pattern.

1B. Label the links q and p with Arabic numerals 1 and 2, respectively. This means that all users prefer link q to link p. The analytical results are shown in Table 7.

The analytical results shown in Table 7 also coincide with the foregoing findings exactly, but the solutions are different from that with preference order 1A. In Table 6, all flow patterns with $f_q < 3.8$ are not at the BRUE state since there does not exist such aspiration level that satisfies the BRUE definition. However, the SO solution is one kind of BRUE state with this preference order.

4.2. Solution method

We now propose a solution method to find the BRUE flow patterns under satisficing rule (i.e., with given aspiration levels). Before designing the algorithm, we investigate the existence and uniqueness of solution, only concerning the networks with parallel links.

Table 6
Analytical results of a two-link network with preference order **1A**.

Link flow	Link travel time	Total time	Min. aspiration level	Solution type
(10, 0)	(22, 0)	220	$\bar{A}_p = 22$	BRUE
(9, 1)	(20, 6)	186	$\bar{A}_p = 20, \bar{A}_q = 6$	BRUE
(8, 2)	(18, 9)	162	$\bar{A}_p = 18, \bar{A}_q = 9$	BRUE
(7, 3)	(16, 12)	148	$\bar{A}_p = 16, \bar{A}_q = 12$	BRUE
(6.2, 3.8)	(14.4, 14.4)	144	$\bar{A}_p = 14.4, \bar{A}_q = 14.4$	BRUE/UE
(6.1, 3.9)	(14.2, 14.7)	143.95		SO
(6, 4)	(14, 15)	144		
(0, 10)	(0, 33)	330		

Table 7
Analytical results of a two-link network with preference order **1B**.

Link flow	Link travel time	Total time	Min. aspiration level	Solution type
(10, 0)	(33, 2)	330	$\bar{A}_q = 33$	BRUE
(9, 1)	(30, 4)	274	$\bar{A}_q = 30, \bar{A}_p = 4$	BRUE
(8, 2)	(27, 6)	228	$\bar{A}_q = 27, \bar{A}_p = 6$	BRUE
(7, 3)	(24, 8)	192	$\bar{A}_q = 24, \bar{A}_p = 8$	BRUE
(6, 4)	(21, 10)	166	$\bar{A}_q = 21, \bar{A}_p = 10$	BRUE
(5, 5)	(18, 12)	150	$\bar{A}_q = 18, \bar{A}_p = 12$	BRUE
(4, 6)	(15, 14)	144	$\bar{A}_q = 15, \bar{A}_p = 14$	BRUE
(3.9, 6.1)	(14.7, 14.2)	143.95	$\bar{A}_q = 14.7, \bar{A}_p = 14.2$	BRUE/SO
(3.8, 6.2)	(14.4, 14.4)	144	$\bar{A}_q = \bar{A}_p = 14.4$	BRUE/UE
(3, 7)	(12, 16)	148		
(0, 10)	(3, 22)	220		

Proposition 1. *Consider an arbitrary user preference order of choosing links. Define a vector of aspiration levels,* $\bar{\mathbf{A}} = (\bar{A}_r)$*. As long as provided appropriate values for components of* $\bar{\mathbf{A}}$*, the BRUE solution exists.*

Proof. With the Definitions 2 and 3 and Assumption 2, take $\bar{A}_{\bar{r}} \geqslant t_{\bar{r}}(d)$, then $f_{\bar{r}}^B = d$. This solution is a BRUE link flow pattern. This is an example of supporting the proposition: as long as provided appropriate values for components of $\bar{\mathbf{A}}$, the BRUE solution exists. □

Proposition 2. *Let* $\mathbf{f} = (f_r)$ *denote a vector of link flows. If* $\bar{A}_{\bar{r}} \geqslant t^U$*, the corresponding* Ω *is then non-empty.*

Proof. Patriksson (1994) showed that, when all link time functions are continuous, the UE exists. Let $\mathbf{f}^U$ be the vector of user equilibrium link flows. From Theorem 3, when $\bar{A}_{\bar{r}} = t^U$, the link flow pattern is a standard UE solution, i.e., $\mathbf{f}^U \in \Omega$. In other words, the UE solution must be contained in the BRUE solution set. Therefore, $\Omega \neq \emptyset$. □

From Assumption 2 and Proposition 2, we can obtain the following proposition straightforward.

Proposition 3. *Under Assumption 2, for a given* $\bar{\mathbf{A}}$*, if the following conditions hold, the BRUE flow pattern exists and is unique,*

$$\bar{A}_{\bar{r}} \geqslant t^U, \tag{10}$$

$$\bar{A}_r \geqslant t_r \geqslant \max_{j \in J}\{A_v^j\}, \quad r \leqslant v, \quad r, v \in R^+, \tag{11}$$

$$\sum_{r \in R} f_r = d, \tag{12}$$

$$f_r \geqslant 0, \quad r \in R, \tag{13}$$

where all inequalities in (10) and (11) become equalities if all users are perfectly rational.

To find the unique BRUE flow pattern, we add the conditions in Proposition 3, the minimal aspiration levels on all routes r with positive flows are equal to the corresponding route travel times. Furthermore, the following proposition gives the elaborate description of Proposition 3.

Proposition 4. *Consider an arbitrary user preference of choosing links, i.e.,* $R = \{1, 2, \ldots, n\}$*. Removing link 1 and the flow* f_1 *from the original network, we denote the UE flow and link time of the new network by* $\mathbf{f}^{1U}$ *and* t^{1U}*, respectively. Similarly, Remove links 1 to r and the flow* $\sum_1^r f_r$ *on them from the original network, and let the UE flow and link time of the new network be* $\mathbf{f}^{rU}$ *and* t^{rU}*, respectively. Under Assumption 2, for a given* $\bar{\mathbf{A}}$*, if the following conditions hold, the BRUE link flow pattern exists and is unique,*

$$\bar{A}_{\bar{r}} \geqslant t^U, \tag{14}$$

$$\bar{A}_r \geqslant \bar{A}_v \geqslant t^{rU}, \quad r \leqslant v, r, v \in R^+, \tag{15}$$

$$\sum_{r \in R} f_r = d, \tag{16}$$

$$f_r \geqslant 0, \quad r \in R, \tag{17}$$

where all inequalities in (14) and (15) become equalities if all users are perfectly rational.

We can stepwise justify Propositions 3 and 4 and then obtain a solution method to solve the BRUE traffic assignment problem. The algorithm is given below.

Step 1. Compute the UE flow $\mathbf{f}^U$ and link time t^U of the original network.

Step 2. For a given $\bar{A}_{\bar{r}} \geqslant t^U$, compute $f_{\bar{r}}^B$ from equation $\bar{A}_{\bar{r}} = t_{\bar{r}}(f_{\bar{r}})$. If $f_{\bar{r}}^B \geqslant d$, then terminate the procedure. The unique BRUE solution is $\mathbf{f} = (0, \ldots, d, \ldots, 0)$. If $f_{\bar{r}}^B < d$, go to Step 3.

Step 3. Link $\bar{r}$ is assigned with flow $f_{\bar{r}}^B$ and the given minimal aspiration level $\bar{A}_{\bar{r}}$. Remove link $\bar{r}$ and the flow $f_{\bar{r}}^B$ from the network. Find the new UE flow $\mathbf{f}^{\bar{r}U}$ and link time $t^{\bar{r}U}$ of the new network. For a given $\bar{A}_v$ ($\bar{A}_{\bar{r}} \geqslant \bar{A}_v \geqslant t^{\bar{r}U}$), compute f_v^B from equation $\bar{A}_v = t_v(f_v)$. If $f_v^B \geqslant d - f_{\bar{r}}^B$, then terminate the procedure and the BRUE solution is $\mathbf{f} = (0, \ldots, f_{\bar{r}}^B, 0, \ldots, d - f_{\bar{r}}^B, 0, \ldots, 0)$. When $f_v^B < d - f_{\bar{r}}^B$, go to Step 4.

Step 4. Suppose links 1 to r are assigned with flows $(f_1^B, \ldots, f_r^B)$ and the given aspiration levels $(\bar{A}_1, \ldots, \bar{A}_r)$. Remove these links and the flows on them from the original network. Find the UE flow $\mathbf{f}^{rU}$ and link time t^{rU} from the new network. For a given $\bar{A}_v$ ($\bar{A}_r \geqslant \bar{A}_v \geqslant t^{rU}$), compute f_v^B from equation $\bar{A}_v = t_v(f_v)$. If $f_v^B \geqslant d - \sum_1^r f_r^B$, then terminate the procedure and the BRUE solution is $\mathbf{f} = (f_1^B, f_2^B, \ldots, f_r^B, d - \sum_1^r f_r^B, 0, \ldots, 0)$. If $f_v^B < d - \sum_1^r f_r^B$, go to Step 5.

Step 5. Let $r = r + 1$ and repeat Step 4 until $f_{r+1}^B \geqslant d - \sum_1^r f_r^B$.

From the above, we find that when appropriate values of aspiration levels are given, the BRUE solution is then unique, because the travel time functions are monotonically increasing and continuously differentiable. Propositions 3 and 4 give the conditions under which the BRUE solution exists and is unique. The minimal aspiration levels for all links are required. Eqs. (10) and (11) (or (14) and (15)) ensure the existence of the BRUE state.

Example 2. To illustrate the solution method, consider a network consisting of three parallel links. The total demand is $d = 10$. The three link travel time functions are, $t_p = 2(f_p)^2 + 2$, $t_q = (f_q)^2 + 3$ and $t_h = (f_h)^2 + 40$, respectively. It is easy to find the UE solution, i.e., $f_p^U = 4.18$, $f_q^U = 5.82$, and $f_h^U = 0$, with minimal equilibrium link travel time $t^U = 36.9$. Note that there exist six possible user preference orders of choosing the three links.

Consider such a preference order that gives links h, q and p with Arabic numerals 1, 2 and 3, respectively. The condition $\overline{A}_{\bar{r}} \geqslant 36.9$ should hold for having the BRUE states. Table 8 gives some BRUE solutions when $\overline{\mathbf{A}}$ takes different values (not all feasible values). From this table, we can verify such a fact that the BRUE solution indeed exists and is unique when $\overline{\mathbf{A}}$ takes an appropriate value. When $\overline{A}_h \geqslant 140$ and $\overline{A}_q = \overline{A}_p = 0$, the BRUE solution always exists and it follows $\mathbf{f} = (10, 0, 0)$ by solving the equation $t_h = (f_h)^2 + 40 = 140$. Under this condition, all 10 users whose aspiration levels are not lower than 140 are satisfied with route h.

When $\overline{A}_h = 121$, by solving the equation $t_h = (f_h)^2 + 40 = 121$, $f_h^B = 9$ can be computed. These 9 users are satisfied with route h. Next we seek the aspiration level conditions for other users. Based on Step 3 of the above solution method, we have $\mathbf{f}^{1U} = (0.27, 0.73)$ and $t^{1U} = 3.07$, then, the aspiration levels of the users who choose route q satisfy the condition $121 \geqslant \overline{A}_q \geqslant 3.07$. Similarly, if we set $\overline{A}_h = 121$, $121 \geqslant \overline{A}_q \geqslant 4$ and $\overline{A}_p = 0$, then, $f_q^B \geqslant d - f_h^B = 1$, the algorithm is terminated and $\mathbf{f} = (9, 1, 0)$ is always the BRUE solution. In other words, under the conditions $\overline{A}_h = 121$ and $121 \geqslant \overline{A}_q \geqslant 4$, 9 users are satisfied with route h, 1 user is satisfied with route q. When $\overline{A}_h = 121$ and $\overline{A}_q = \overline{A}_p = 3.07$, the flow pattern $f_h^B = 9$, $f_q^B = 0.27$ and $f_p^B = 0.73$ can be computed, then, $\mathbf{f} = (9, 0.27, 0.73)$ is the BRUE solution.

Similar to the above analysis, when $\overline{A}_h = 104$, $f_h^B = 8$ can be computed. We then have $\mathbf{f}^{1U} = (1, 1)$ and $t^{1U} = 4$, then, $104 \geqslant \overline{A}_q \geqslant 4$. If we set $\overline{A}_h = 104$, $104 \geqslant \overline{A}_q \geqslant 7$ and $\overline{A}_p = 0$, then, $f_q^B \geqslant d - f_h^B = 2$, $\mathbf{f} = (8, 2, 0)$ is always the BRUE solution. When $\overline{A}_h = 104$ and $\overline{A}_q = \overline{A}_p = 4$, $f_h^B = 8$ and $f_q^B = f_p^B = 1$ can be computed, then, $\mathbf{f} = (8, 1, 1)$ is the BRUE solution.

When $\overline{A}_h \leqslant t_h(0) = 40$, this means no users choose link h at BRUE state, then, we have $\overline{A}_q \leqslant \overline{A}_p \leqslant 40$, $f_h^B = 0$, $f_q^B + f_p^B = 10$, $\mathbf{f}^{1U} = (5.82, 4.18)$ and $t^{1U} = 36.9$. If we set $\overline{A}_q = 40$ and $40 \geqslant \overline{A}_p \geqslant 32.69$, then, $f_q^B = 6.08$, $f_p^B \geqslant d - f_q^B = 3.92$, $\mathbf{f} = (0, 6.08, 3.92)$ is always the BRUE solution. When $\overline{A}_q = \overline{A}_p = 36.9$, $f_q^B = 5.82$ and $f_p^B = 4.18$ can be computed, then, $\mathbf{f} = (0, 5.82, 4.18)$ is the BRUE solution.

When $\overline{A}_q = 36$, there doesn't exist any BRUE solution since this aspiration level is less than $t^U = 36.9$.

5. Extension to general networks

In this section we extend our work to general networks. The definition of the BRUE state in general networks can be formulated as follows:

$$T_{rw} \leqslant \overline{A}_{rw} = \min_{j \in J}\{A_{rw}^j\}, \quad r \in R_w^+, \quad w \in W, \tag{18}$$

$$\sum_{r \in R_w} f_{rw} = d_w, \quad w \in W, \tag{19}$$

$$f_{rw} \geqslant 0, \quad r \in R_w, \quad w \in W. \tag{20}$$

Table 8
BRUE solutions when $\overline{\mathbf{A}}$ takes different values in Example 2.

$\overline{A}_h$	$\overline{A}_q$	$\overline{A}_p$	**F**
140	0	0	(10, 0, 0)
121	121	0	(9, 1, 0)
121	4	0	(9, 1, 0)
121	3.07	3.07	(9, 0.27, 0.73)
104	104	0	(8, 2, 0)
104	7	0	(8, 2, 0)
104	4	4	(8, 1, 1)
89	89	0	(7, 3, 0)
89	12	0	(7, 3, 0)
89	5.7	5.7	(7, 1.64, 1.36)
40	40	32.69	(0, 6.08, 3.92)
0	37.18	36.51	(0, 5.85, 4.15)
0	36.9	36.9	(0, 5.82, 4.18)
0	36	36	–

Similar to the analysis in the parallel-link networks, some extended properties in general networks based on the Assumption 1 can be derived.

5.1. Properties of the BRUE state

For any OD pair $w \in W$, consider an arbitrary user preference order of routes, without loss of generality, we label the order by Arabic numerals, $1, 2, \ldots, n$, and denote the set by $R_w = \{1, 2, \ldots, n\}$, $w \in W$. The fundamental assumption, i.e., Assumption 1, can be replaced by the following Assumption 3.

Assumption 3. For all r and v, $r \leqslant v, r, v \in R_w, w \in W$, users prefer to choose route r than route v if both routes are acceptable.

Lemma 3. *With Assumption 3, at BRUE state, we have* $\max_{j \in J}\{A^j_{vw}\} \leqslant \min_{j \in J}\{A^j_{rw}\}$ *and* $T_{vw} \leqslant T_{rw}$, *for all* r *and* v, $r \leqslant v, r, v \in R^+_w, w \in W$.

Theorem 1 (*extended*). *A link-based flow distribution is at route preference-based BRUE state, if and only if the following conditions are satisfied for all r and v,* $r \leqslant v, r, v \in R^+_w, w \in W$,

$$\overline{A}_{rw} \geqslant T_{rw} \geqslant \max_{j \in J}\{A^j_{vw}\}, \tag{21}$$

$$\sum_{r \in R_w} f_{rw} = d_w, \tag{22}$$

$$x_a = \sum_{r \in R_w} f_{rw}\delta_{ar}, \quad a \in L. \tag{23}$$

We call conditions (21)–(23) as boundedly rational user equilibrium conditions.

Note that Lemma 3 and Theorem 1 (extended) can be easily derived based on Lemma 1, Lemma 2 and Theorem 1 for parallel-link networks.

Theorem 2 (*extended*). *Denote the BRUE travel time on route r for OD pair* $w \in W$ *by* T^B_{rw}, $r \in R^+_w$, *the UE travel time on route r by* T^U_{rw}, *the minimal Arabic numeral in the set* R^+_w *by* $\bar{r}$. *Then, at BRUE state, there always exists at least one OD pair such that the condition* $\overline{A}_{\bar{r}w} \geqslant T^B_{\bar{r}w} \geqslant T^U_{\bar{r}w}$ *holds.*

Theorem 3 (*extended*). *Let* T^U_w *denote the minimal route travel time for OD pair* $w \in W$ *under UE conditions. When* $\overline{A}_{\bar{r}w} = T^B_{\bar{r}w} = T^U_w$ *hold for all OD pair* $w \in W$, *the link-based BRUE solution reduces to the standard link-based UE solution.*

For more details of the proof for Theorems 2 and 3 (extended), readers can refer to the proof for parallel-link networks in Section 4. Both theorems can be proved by the contradiction method.

5.2. Solution method

We now propose a solution method to find the link-based BRUE flow patterns under satisficing rule in general networks (i.e., with given aspiration levels). Before designing the algorithm, we investigate the existence and uniqueness of solution.

Proposition 1 (*extended*). *Consider an arbitrary combined user preference order of choosing routes. Define a vector of aspiration levels,* $\overline{\mathbf{A}} = (\overline{A}_{rw}), w \in W$. *As long as provided appropriate values for components of* $\overline{\mathbf{A}}$, *the BRUE solution exists.*

Note that for general networks, there exist many OD pairs and overlapping links, we have to consider the combined route preference orders. Clearly, for all OD pairs, when the conditions $\overline{A}_{rw} = T_{rw}(\sum_w d_w), w \in W$, are given, the BRUE solution exists.

Proposition 2 (*extended*). *If* $\overline{A}_{\bar{r}w} \geqslant T^U_w, w \in W$, *the corresponding feasible set of link flows X is then non-empty.*

Note that the proof of Proposition 2 (extended) is similar to that of Proposition 2.

Proposition 3 (*extended*). *With Assumption 3 and linear link travel time functions, for a given* $\overline{\mathbf{A}}$, *if the following conditions hold, the link-based BRUE flow pattern exists and is unique,*

$$\overline{A}_{\bar{r}w} \geqslant T^U_{\bar{r}w}, \quad \forall w, \tag{24}$$

$$\overline{A}_{rw} = T_{rw} \geqslant \max_{j \in J}\{A^j_{vw}\}, \quad r \leqslant v, \quad r, v \in R^+_w, \quad \forall w, \tag{25}$$

$$\sum_{r \in R_w} f_{rw} = d_w, \forall w, \tag{26}$$

$$f_{rw} \geqslant 0, \quad \forall r, w, \tag{27}$$

$$x_a = \sum_{r \in R_w} f_{rw}\delta_{ar}, \quad a \in L. \tag{28}$$

To ensure the unique link-based BRUE flow pattern, we adopt the assumption and linear link travel time functions stated in Proposition 3 (extended) for general networks.

The solution method to solve the BRUE traffic assignment problem is given below.

Step 1. Solve a variant UE problem MP1

Given the conditions $\overline{A}_{\bar{r}w} = T_{\bar{r}w} \geqslant T^U_{\bar{r}w}, w \in W$, where $\bar{r}$ is the minimal Arabic numeral in the set R^+_w, consider the following mathematical programming (MP1):

$$\min Z(\mathbf{x}) = \sum_a \int_0^{x_a} t_a(x)\mathrm{d}x \tag{29}$$

subject to

$$\sum_{r \in R_w} f_{rw} = d_w, \quad \forall w, \tag{30}$$

$$T_{\bar{r}w} = \overline{A}_{\bar{r}w}, \quad \forall w, \tag{31}$$

$$f_{rw} \geqslant 0, \quad \forall r, w, \tag{32}$$

$$x_a = \sum_{r \in R_w} f_{rw}\delta_{ar}, \quad a \in L. \tag{33}$$

This is a simple variant of the standard UE model (Beckmann et al., 1956) with addition of the aspiration level satisfaction conditions (31). Since the problem is a convex programming problem with linear constraints, for an optimum solution $\mathbf{f}^{\bar{r}*} = (f^*_{rw}, r \in R_w, w \in W)$ and $\mathbf{x}^{\bar{r}*} = (x^*_a, a \in L) \in X$, there exist the Lagrange multipliers $\boldsymbol{\mu} = (\mu_w, w \in W)$ and $\boldsymbol{\eta} = (\eta_w, w \in W)$ associated with the path flow conservation constraints (30) and the aspiration level satisfaction conditions (31) respectively, such that the following first-order optimality conditions hold:

$$\left(\sum_a t_a(x^*_a)\delta_{ar} - \mu_w + \lambda_w\right) f^*_{rw} = 0, \quad r \in R_w \setminus \{\bar{r}\}, \quad \forall w \tag{34}$$

$$\sum_a t_a(x^*_a)\delta_{ar} - \mu_w + \lambda_w \geqslant 0, \quad f^*_{rw} \geqslant 0, r \in R_w \setminus \{\bar{r}\}, \quad \forall w \tag{35}$$

$$\sum_{r \in R_w} f_{rw} = d_w, \forall w \tag{36}$$

$$\overline{A}_{\bar{r}w} = T_{\bar{r}w}, \quad \forall w \tag{37}$$

where $\lambda_w = \eta_w \cdot k_{\bar{r}}$, $k_{\bar{r}}$ is constant. $R_w \setminus \{\bar{r}\}$ is the set of routes excluding $\{\bar{r}\}$ between OD pair w.

It is evident that the first-order optimality conditions (34)–(37) are equivalent to the UE conditions for routes r, $r \in R_w \setminus \{\bar{r}\}, \forall w$, and the aspiration level satisfaction conditions (31) for route $\bar{r}$. For route $r \in R_w \setminus \{\bar{r}\}, \forall w$, the minimal equilibrium route travel time is $T^*_{\bar{r}w} = \mu_w - \lambda_w \leqslant \mu_w$, which is not higher than the minimal equilibrium route travel time under the standard UE conditions. The reason is that there are more users preferring to choose the route $\bar{r}$. Since both the objective function and the constraints of the problem (29)–(33) are convex in x_a, the link flow pattern at equilibrium, x^*_a, is unique. However, as in the standard UE model, the path flow pattern, f^*_{rw}, at equilibrium is usually not unique.

Step 2. Define another variant UE problem MP2

We denote the new equilibrium route travel time under the given conditions $\overline{A}_{\bar{r}w} = T_{\bar{r}w} \geqslant T^U_w, w \in W$ as T^{1U}_{rw}, then we have, at BRUE state, $\overline{A}_{\overline{v}w} = T_{\overline{v}w} \geqslant T^{1U}_{\overline{v}w}, r \in R_w \setminus \{\bar{r}\}, w \in W$.

Given the conditions $\overline{A}_{\bar{r}w} \geqslant \overline{A}_{\overline{v}w} = T_{\overline{v}w} \geqslant T^{1U}_{\overline{v}w}, \bar{v} \in R_w \setminus \{\bar{r}\}, w \in W$, define a new mathematical programming (MP2): (29), subject to (30)–(33) and the following aspiration level satisfaction conditions

$$\overline{A}_{\overline{v}w} = T_{\overline{v}w}, \bar{v} \in R_w \setminus \{\bar{r}\}, \quad w \in W. \tag{38}$$

Step 3. Repeat to solve MP2

Each time, new aspiration level satisfaction conditions are added, until all these conditions are considered. Then, we get the link-based BRUE flow pattern.

It should be particularly pointed out that the conventional Frank-Wolf algorithm (Sheffi, 1985) can be easily applied to solve the above mathematical programming problems.

In summary, an enumeration method is used to find the link-based BRUE solution in this paper. Developing an analytical method is our future research topic.

Example 3. Consider a network shown in Fig. 4 which contains one origin–destination pair, three nodes and four links. Node 1 is the origin, and Node 3 is the destination. The total demand is 10. Path 1 (P1) consists of link 1 and link 3, path 2 (P2) link 1 and link 4, path 3 (P3) link 2 and link 3, path 4 (P4) link 2 and link 4. The link travel time functions are $t_1 = x_1 + 1$, $t_2 = x_2 + 4$, $t_3 = x_3 + 2$, and $t_4 = x_4 + 4$. The unique UE link flow solution is $x_1^U = 6.5$, $x_2^U = 3.5$, $x_3^U = 6$, $x_4^U = 4$, and the equilibrium path travel time $T^U = 15.5$.

Note that there exist 24 possible user preference orders of choosing the four routes. As an illustration of the study, we take the route preference that is ordered by increasing free flow route travel times, i.e. P1, P2, P3, P4.

From Theorem 2 (extended), at BRUE state, we have $\overline{A}_{P1} \geqslant T_{P1}^B \geqslant T_{P1}^U$. Next we give the values of aspiration levels for finding the link-based BRUE solution.

Given the condition $\overline{A}_{P1} = 19 > T_{P1}^U = 15.5$, by solving the MP1, we have the unique link flow pattern: $x_1^{1*} = 8$, $x_2^{1*} = 2$, $x_3^{1*} = 8$, $x_4^{1*} = 2$. The equilibrium route travel times are: $T_{P1}^{1U} = 19$, $T_{P2}^{1U} = 15$, $T_{P3}^{1U} = 16$, and $T_{P4}^{1U} = 12$, respectively. Clearly, given the condition $12 \leqslant \overline{A}_{P4} \leqslant 15$, these users are satisfied with path 4. Then, the aspiration level satisfaction conditions are satisfied. In all, given the conditions $\overline{A}_{P1} = 19$ and $12 \leqslant \overline{A}_{P4} \leqslant 15$, we can get the unique link-based BRUE solution: $x_1^{1*} = x_3^{1*} = 8$, $x_2^{1*} = x_4^{1*} = 2$. In addition, when given the condition $15 \leqslant \overline{A}_{P2} \leqslant 19$, e.g., $\overline{A}_{P2} = 18$, the MP1 with this condition added can be solved to get the unique solution $\mathbf{x}^{2*}$.

Example 3 shows that if the aspiration level vector takes an appropriate value, the link-based BRUE solution exists and is unique.

Example 4. In Fig. 5, the road network contains four nodes and five directed links. There are two OD pairs denoted by $a = (1,4)$ and $b = (2,4)$, respectively. The OD demands are $d_a = 20$ and $d_b = 30$. There are two paths connecting each OD pair. For OD pair a, path 1 (P1) covers link 1 only and path 2 (P2) is composed of links 2 and 3. For OD pair b, path 3 (P3) covers links 3 and 4, and path 4 (P4) includes link 5 only. The link time functions are given by $t_1 = 20 + 2x_1$, $t_2 = x_2$, $t_3 = x_3$, $t_4 = 20 + x_4$, $t_5 = 2x_5$. The unique UE link flow solution is $x_1^U = 20/3$, $x_2^U = 40/3$, $x_3^U = 20$, $x_4^U = 20/3$, and $x_5^U = 70/3$. The equilibrium path travel times are $T_a^U = 100/3$ and $T_b^U = 140/3$.

We take the route preference order as an example: for OD pair a, users prefer P1 to P2; for OD pair b, users prefer P3 to P4.

From Theorem 2 (extended), at BRUE state, we have $\overline{A}_{P1} \geqslant T_{P1}^B \geqslant T_a^U$ and $\overline{A}_{P3} \geqslant T_{P3}^B \geqslant T_b^U$. Next, we give the values of aspiration levels for finding the link-based BRUE solution.

Given the condition $\overline{A}_{P1} = 60 > T_a^U = 100/3$, by solving MP1, we have the unique link flow pattern: $x_1^{1*} = 20$, $x_2^{1*} = 0$, $x_3^{1*} = 40/3$, $x_4^{1*} = 40/3$, and $x_5^{1*} = 50/3$. The route travel times are: $T_{P1}^{1U} = 60$, $T_{P2}^{1U} = 40/3$, and $T_{P3}^{1U} = T_{P4}^{1U} = 100/3$. Given the second condition $\overline{A}_{P3} = 80 > T_{P3}^{1U} = 100/3$, by solving MP2, we have the unique link flow pattern: $x_1^{2*} = 20$, $x_2^{2*} = 0$, $x_3^{2*} = 30$, $x_4^{2*} = 30$, $x_5^{2*} = 0$. The route travel times are: $T_{P1}^{2U} = 60$, $T_{P2}^{2U} = 30$, $T_{P3}^{2U} = 80$, $T_{P4}^{2U} = 0$. In all, given the aspiration level conditions $\overline{A}_{P1} = 60$ and $\overline{A}_{P3} = 80$, we get the unique link-based BRUE solution: $x_1^B = 20$, $x_2^B = 0$, $x_3^B = x_4^B = 30$, $x_5^B = 0$.

If given the second condition $\overline{A}_{P3} = 60 > T_{P3}^{1U} = 100/3$, by solving MP2, we have the unique link flow pattern: $x_1^{2*} = 20$, $x_2^{2*} = 0$, $x_3^{2*} = 20$, $x_4^{2*} = 20$, $x_5^{2*} = 10$. The route travel times are: $T_{P1}^{2U} = 60$, $T_{P2}^{2U} = 20$, $T_{P3}^{2U} = 60$, $T_{P4}^{2U} = 20$. Then, given the aspiration level conditions $\overline{A}_{P1} = 60$, $\overline{A}_{P3} = 60$ and $20 \leqslant \overline{A}_{P4} \leqslant 60$, we get the unique link-based BRUE solution: $x_1^B = 20$, $x_2^B = 0$, $x_3^B = x_4^B = 20$, $x_5^B = 10$.

Given the conditions $\overline{A}_{P1} = 50 > T_a^U = 100/3$, by solving MP1, we have the unique link flow pattern: $x_1^{1*} = 15$, $x_2^{1*} = 5$, $x_3^{1*} = 55/4$, $x_4^{1*} = 35/4$, $x_5^{1*} = 85/4$. The route travel times are: $T_{P1}^{1U} = 50$, $T_{P2}^{1U} = 75/4$, $T_{P3}^{1U} = T_{P4}^{1U} = 85/2$. Given the second con-

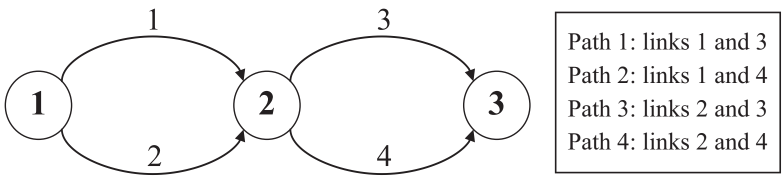

Fig. 4. A network having three nodes and four links.

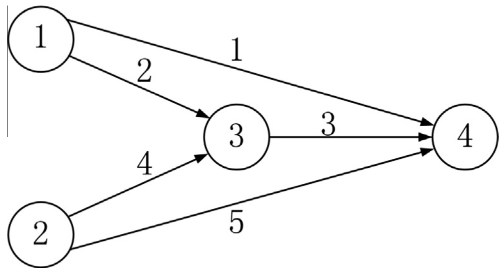

Fig. 5. A network with two OD pairs.

dition $\overline{A}_{P3} = 85 > T_{P3}^{1U} = 85/2$, by solving MP2, we have the unique link flow pattern: $x_1^{2*} = 15$, $x_2^{2*} = 5$, $x_3^{3*} = 35$, $x_4^{2*} = 30$, $x_5^{2*} = 0$. The route travel times are: $T_{P1}^{2U} = 50$, $T_{P2}^{2U} = 40$, $T_{P3}^{2U} = 85$, $T_{P4}^{2U} = 0$. In all, given the aspiration level conditions $\overline{A}_{P1} = 50$, $40 \leqslant \overline{A}_{P2} \leqslant 50$ and $\overline{A}_{P3} = 85$, we get the unique link-based BRUE solution: $x_1^B = 15$, $x_2^B = 5$, $x_3^B = 35$, $x_4^B = 30$, $x_5^B = 0$.

Until given the aspiration level conditions $\overline{A}_{P1} = \overline{A}_{P2} = 100/3$, $\overline{A}_{P3} = \overline{A}_{P4} = 140/3$, we can get the unique link-based BRUE solution: $x_1^B = 20/3$, $x_2^B = 40/3$, $x_3^B = 20$, $x_4^B = 20/3$, $x_5^B = 70/3$. Note that this BRUE solution is just the standard UE solution.

In summary, for Example 4, if the aspiration level vector meets certain conditions, the link-based BRUE solution exists and is unique.

6. Conclusions

For many years after Simon's initial work of bounded rationality, it was recognized that this topic was of great importance, but the lack of a formal approach impeded its progress (Aumann, 1997). In this paper, we investigated the boundedly rational route choice behavior under the Simon's satisficing rule by introducing the concept of aspiration level. The strict assumption that all users have the same route preference order was used throughout the paper. Some properties of the BRUE state were analytically examined by looking into the definitions and assumptions. It was shown that if the aspiration level vector takes an appropriate value, the BRUE solution exists and is unique. Furthermore, it can be obtained by using the proposed solution method.

The value of our study lies in the starting of researching a new type of traffic assignment. The analysis can be extended along several lines, for example, to consider the demand elasticity, aspiration level adjustment and nonlinear link travel time function in general networks, and so on. If so, more complex models must be developed, but the analyses and explanations would be yet more illuminating. In addition, conducting other kinds of experimental studies, developing an analytical method to derive the values of aspiration levels and investigating the boundedly rational route choice behavior under various congestion pricing schemes for realizing the SO flow pattern, are also meaningful.

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