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Second best toll pricing within the framework of bounded rationality

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ABSTRACT

The network design problem is usually formulated as a bi-level program, assuming the user equilibrium is attained in the lower level program. Given boundedly rational route choice behavior, the lower-level program is replaced with the boundedly rational user equilibria (BRUE). The network design problem with boundedly rational route choice behavior is understudied due to non-uniqueness of the BRUE. In this study, thus, we mainly focus on boundedly rational toll pricing (BR-TP) with affine link cost functions. The topological properties of the lower level BRUE set are first explored. As the BRUE solution is generally non-unique, urban planners cannot predict exactly which equilibrium flow pattern the transportation network will operate after a planning strategy is implemented. Due to the risk caused by uncertainty of people's reaction, two extreme scenarios are considered: the traffic flow patterns with either the minimum system travel cost or the maximum, which is the "risk-prone" (BR-TP-RP) or the "risk-averse" (BR-TP-RA) scenario respectively. The upper level BR-TP is to find an optimal toll minimizing the total system travel cost, while the lower level is to find the best or the worst scenario. Accordingly BR-TP can be formulated as either a min–min or a min–max program. Solution existence is discussed based on the topological properties of the BRUE and algorithms are proposed. Two examples are accompanied to illustrate the proposed methodology.

1. Introduction and motivation

1.1. Network design problem

The network design problem (NDP) is to relieve road network congestion by determining how much toll pricing should be charged on congested links, by selecting which links need to be extended, or by considering where new links should be added, given the traffic demand for each origin-destination pair. There have been many studies on transportation network design and interested readers can refer to Yang and Bell (1998) for detailed reviews of the NDP literature.

Among these studies, NDP is usually formulated as a bi-level program: the upper level is the decision made to either execute toll pricing, enhance capacities of the selected links, or add new links to the existing road network, while the lower level is an equilibrium problem, describing how drivers will distribute among the new network topology. Accordingly, the decision variables in the upper level represent whether to charge tolls, to widen a link, or to build a new road, and can be either continuous

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or discrete. The lower level problem is the link or path flow distribution based on some traffic assignment principle. When Wardrop's first principle (Wardrop, 1952) is adopted, the user equilibrium (UE) is obtained in the lower level problem.

The above bi-level NDP can be also regarded as a Stackelberg game (i.e., leader-follower game Yang and Bell (1998)). The leader is the decision-maker managing the upper level variables, while individual drivers are followers who play the non-cooperative Nash game on the fixed topology of the road network. In other words, provided the upper level decision, each driver will response accordingly by playing a Nash game and choose her own route in response to the upper level decision.

1.2. Non-uniqueness of the lower level program

If the lower level problem has only one equilibrium (i.e., each driver has a single action strategy in response to a network change plan), planners can make an exact prediction of how drivers will react to a certain transportation plan. However, if there exist multiple equilibria in the lower level problem (i.e., each driver has multiple action strategies), planners do not know exactly how drivers will behave, then solving a standard bi-level program may not achieve the expected goal of improving efficiency of the transportation system. Because the bi-level program "implicitly assumes that by applying the obtained toll, the resulting UE flow pattern is exactly what is predicted by the model, and thus the desired system objective can be achieved" (Ban et al., 2009). Ban et al. (2009) also proposed two concepts: the "predicted" and the "realized" equilibrium, to further explain consequences resulting from implementing a toll obtained from the bi-level program when the lower level problem does not have a unique solution. The former represents the equilibrium obtained from solving the bi-level program, while the latter means the actual equilibrium where the system runs after a network plan is implemented. As drivers play a non-cooperative Nash game in the lower level, there is no way to control which routes they can take. Therefore, the "realized" equilibrium usually deviates from the "predicted" one.

Due to the gap between the "predicted" and the "realized" equilibria, planners face risk or uncertainty while making network plans. There are three attitudes towards risk encountered in the NDP (Ban et al., 2009): risk-prone, risk-neutral and risk-averse. "Risk prone" assumes that planners are optimistic and believe the realized equilibrium will be attained at the most efficient flow pattern, i.e., the one with the lowest system travel time; then they try to find a toll which can improve system efficiency at this equilibrium. In contrast, "risk averse" represents the attitude that planners try to find a toll which can improve system efficiency by assuming the realized equilibrium is attained at the least efficient flow pattern, i.e., the one with the largest system travel time. Planners are "risk-neutral" when they assume each flow pattern has certain probability to occur and they aim to find a toll which can improve system efficiency at the expected equilibrium (Ban et al., 2013).

1.3. NDP with boundedly rational user equilibria in the lower level

Regarding the lower level traffic assignment problem, most NDP literature assumes drivers are perfectly rational (PR) and its resultant equilibrium follows Wardrop's first principle (e.g., Leblanc 1975; Magnanti and Wong 1984; Yang and Bell 1998). However, many empirical studies using simulation experiments (Nakayama et al., 2001), stated preference surveys (Avineri and Prashker, 2004), and GPS vehicle trajectory data (Morikawa et al., 2005; Zhu, 2011) showed that in reality, this assumption is too restrictive and drivers do not always choose the shortest paths. Thus, many other behavioral models have been developed to relax the PR assumption for the traffic assignment problem (see Xu et al. (2011) for the classification of these models).

One way to relax the PR assumption is to assume people are boundedly rational (BR) in the route-choice process. Under the BR route choice paradigm, each traveller has a set of paths to choose instead of travelling only on the shortest paths. The paths which are within a threshold from the shortest path cost can be taken, where this threshold is called "indifference band." There have been many empirical and experimental studies to establish validity of bounded rationality and interested readers can refer to the associated literature (Di, 2014; Di et al., 2014a; Di and Liu, 2015; Di et al., 2015a, 2015b; Hu and Mahmassani, 1997; Jan et al., 2000; Mahmassani and Chang, 1987; Morikawa et al., 2005; Ramming, 2001).

Within the BR framework, the equilibrium obtained from the lower level traffic assignment problem is not UE any more; instead, it is boundedly rational user equilibria (BRUE). Usually the BRUE is not unique and the set consisting of all BRUE flow patterns is called the 'BRUE flow set'. This set has been proven to be non-convex (Di et al., 2013; Lou et al., 2010). Therefore it is quite challenging to solve the NDP with boundedly rational user equilibria (BR-NDP). Within the 'risk averse' scheme, Lou et al. (2010) minimized the 'worst case' scenario (i.e., assuming the network runs least efficiently in terms of the total system travel time) among all BRUE solutions.

1.4. Contribution of this paper

Because the network design problem with bounded rationality behavioral assumption (BR-NDP) is a challenging problem and there are not many established studies, we will mainly focus on the continuous version of the NDP: the second best toll pricing with boundedly rational user equilibria (BR-TP), given the affine link cost functions. The second best toll pricing is to determine an optimal toll vector for a subset of links so that the transportation network can run more efficiently (Ban et al., 2009).

Some existing literature proposed methodologies of solving BRUE (Di et al., 2013; Lou et al., 2010), however, there does not exist any study on the topological properties of BRUE sets, which can facilitate applications of BR-NDP. This paper serves as the first to explore topology of BRUE sets.

In congestion pricing, Lou et al. (2010) formulated a mathematical program with equilibrium constraint (MPEC) to solve the risk-averse toll within the framework of bounded rationality. Due to non-convexity and violation of the Mangasarian–Fromovitz constraint qualification (MFCQ), the solution existence condition was not easy to establish and it was difficult to solve the risk-averse toll. Some heuristic algorithm was employed to obtain an optimal toll.

In this study, we will take advantage of the topological structure of the BRUE set revealed in Di et al. (2013) and establish solution existence conditions by decomposing BR-TP into multiple mathematically tractable subproblems. We will also develop a heuristic algorithm of solving risk-averse BR-TP in a small network. Such analysis heavily relies on the topological properties of the BRUE set and it is non-trivial to derive topological properties of a nonconvex set. This study serves as the first to characterize these properties of the BRUE set for network design problems. Han et al. (2015) briefly discussed the topology of discrete time boundedly rational dynamic user equilibrium (BR-DUE) solution sets and showed that these sets are compact and there always exists a connected but possibly nonconvex subset which can be generated from a single solution. However, the purpose of that paper was to characterize BR-DUE solution sets instead of network design.

As the lower level of BR-NDP contains non-unique equilibria, we will first propose the methodology of constructing the BRUE set in Section 2. In Section 3, the topological properties of the tolled BRUE set are studied to facilitate the solution existence analysis of the boundedly rational toll pricing. In Section 4, the formulation of toll pricing under bounded rationality is proposed. The “risk-prone” and “risk-averse” toll pricing schemes are proposed. In Section 5, solution existence of the risk-prone and risk-averse optimal tolls is discussed. Two examples are given to illustrate the properties of the optimal tolls under difference toll schemes in Section 6. Conclusions and future work are followed in Section 7.

2. $\boldsymbol{\varepsilon}$-BRUE solution set

The traffic network is represented by a directed graph that includes a set of consecutively numbered nodes, $\mathcal{N}$, and a set of consecutively numbered links, $\mathcal{L}$. Let $\mathcal{W}$ denote the O-D pair set connected by a set of simple paths (composed of a sequence of distinct nodes), $\mathcal{P}^w$, through the network. The traffic demand between each OD pair is $d^w > 0$. Let $f_i^w \geqslant 0$ denote the flow on path $i \in \mathcal{P}^w$ for OD pair w, then the path flow vector is $\mathbf{f} = \{f_i^w\}_{i\in\mathcal{P}^w}^{w\in\mathcal{W}}$. For the fixed travel demand case, the feasible path flow set is to assign the traffic demand on the feasible paths: $\mathcal{F} \triangleq \{\mathbf{f} : \mathbf{f} \geqslant \mathbf{0}, \sum_{i\in\mathcal{P}^w} f_i^w = d^w, \forall w \in \mathcal{W}\}$. Denote x_a as the link flow on link a, then the link flow vector is $\mathbf{x} = \{x_a\}_{a\in\mathcal{L}}$. Let $\delta_{ai}^w = 1$ if link a is on path i connecting OD pair w, and 0 if not; then $\Delta \triangleq \{\delta_{ai}^w\}_{a\in\mathcal{L}, i\in\mathcal{P}^w}^{w\in\mathcal{W}}$, denotes the link-path incidence matrix. So $\mathbf{x} = \Delta\mathbf{f}$.

Each link $a \in \mathcal{L}$ is assigned a cost function which is usually monotonously increasing with respect to the link flow, written as $t(\mathbf{x})$. Denote $C_i^w(\mathbf{f})$ as the path cost on path $i \in \mathcal{P}^w$ for OD pair w, then the path cost vector $C(\mathbf{f}) \triangleq \{C_i^w(\mathbf{f})\}_{i\in\mathcal{P}^w}^{w\in\mathcal{W}}$. So $C(\mathbf{f}) = \Delta^T t(\mathbf{x})$ under the additive path cost assumption.

The link set $\mathcal{L}$ can be divided into two disjoint subsets: the link set without tolls charged, $\mathcal{A}$, and the link set with tolls charged, $\mathcal{T}$. Then $\mathcal{L} = \mathcal{A} \cup \mathcal{T}$. Denote $y_a \geq 0$ as the toll charged on link $a \in \mathcal{T}$ and $\mathbf{y}_a = \{y_a\}_{a\in\mathcal{T}}$ as the link toll vector. Given a “value of time” parameter θ, the tolled link cost is computed as $t(\mathbf{y}_a, \mathbf{x}) = t(\mathbf{x}) + \frac{\mathbf{y}_a}{\theta}$. Also, denote the toll charged along the path $i \in \mathcal{P}^w$ as y_i and $\mathbf{y}_p = \{y_i\}_{i\in\mathcal{P}^w}^{w\in\mathcal{W}}$ as the toll vector along the paths. Then the path toll can be computed as the sum of link tolls along the path i: $y_i = \sum_{a\in\mathcal{T}} \delta_{ai} y_a$. The feasible path toll set is thus defined as:

$$\mathcal{Y} \triangleq \{\mathbf{y} : 0 \leqslant y_i \leqslant n_p \bar{y}, i \in \mathcal{P}^w\}, \tag{2.1}$$

where n_p is the number of links along the path i and $\bar{y}$ is the upper bound of a path toll. The tolled path cost vector is computed as $C(\mathbf{y}_p, \mathbf{f}) = \Delta^T t(\mathbf{y}_a, \mathbf{x}) = \Delta^T\left[t(\mathbf{x}) + \frac{\mathbf{y}_a}{\theta}\right] = C(\mathbf{f}) + \frac{\mathbf{y}_p}{\theta}$.

In this study, we employ the affine cost function $t(\mathbf{x}) = A\mathbf{x} + b$, where $A \succ 0$ and $\succ 0$ indicates that a matrix is positive definite. Then $t(\mathbf{x})$ is monotone and the existence of UE and BRUE is ensured (Di et al., 2013). The path cost function $C(\mathbf{f}) = \Delta^T A \Delta \mathbf{f} + \Delta^T b \triangleq Q\mathbf{f} + q$, where $Q \triangleq \Delta^T A \Delta$ and $q \triangleq \Delta^T b$. As defined, Q is symmetric because A is so. Moreover, it is positive semi-definite (i.e., $Q \succeq 0$) because $A \succ 0$ and Δ may not have a full column rank. The dimensions of Q and q are: $Q \in \mathbb{R}^{|\mathcal{P}|\times|\mathcal{P}|}$, $q \in \mathbb{R}^{|\mathcal{P}|}$, respectively, where $|\cdot|$ represents the cardinality of a set and $|\mathcal{P}|$ denotes the total number of paths in a network. Then the tolled path cost is: $C(\mathbf{y}_p, \mathbf{f}) = Q\mathbf{f} + q + \frac{\mathbf{y}_p}{\theta}$. In this paper we assume that $\theta = 1$ without loss of generality.

Definition 2.1. For a given $\varepsilon \geq 0$, a feasible path flow vector $\mathbf{f} \in \mathcal{F}$ is said to be a $\boldsymbol{\varepsilon}$-BRUE path flow pattern, denoted by $\mathbf{f}_{BRUE}^{\varepsilon}$, if

$$f_i^w > 0 \Rightarrow C_i^w(\mathbf{f}) \leqslant \min_{j\in\mathcal{P}^w} C_j^w(\mathbf{f}) + \varepsilon^w, \forall i \in \mathcal{P}^w, \forall w \in \mathcal{W}$$

where $\boldsymbol{\varepsilon} = \{\varepsilon^w\}^{w\in\mathcal{W}}$ is a known positive vector, and each of its element ε^w is the ‘indifference band’ for OD pair w, representing a path cost threshold beyond which travelers are indifferent to.

Usually an $\boldsymbol{\varepsilon}$-BRUE path flow distribution is non-unique given the link cost function is monotonously increasing with respect to the link flow (Di et al., 2013). Denote a set containing all path flow patterns satisfying Definition (2.1) as the $\boldsymbol{\varepsilon}$-BRUE path flow solution set:

$$\mathcal{F}_{BRUE}^{\boldsymbol{\varepsilon}} \triangleq \{\mathbf{f} \in \mathcal{F} : f_i^w > 0 \Rightarrow C_i^w(\mathbf{f}) \leqslant \min_{j\in\mathcal{P}^w} C_j^w(\mathbf{f}) + \varepsilon^w, \forall i \in \mathcal{P}^w, \forall w \in \mathcal{W}\}. \tag{2.2}$$

2.1. A BRUE solution

Di et al. (2013) showed that one BRUE flow pattern can be solved by a nonlinear complementarity problem (BRUE-NCP) and we will revisit it here.

Proposition 2.1 (BRUE NCP (Di et al., 2013)). *Given $\varepsilon^w (w \in \mathcal{W})$, and some $\boldsymbol{\rho} = (\rho_i^w)_{i \in calP^w}^{w \in \mathcal{W}}$, where $0 \leqslant \rho_i^w \leqslant \varepsilon^w$. A feasible path flow vector $\mathbf{f} \in \mathcal{F}$ is a $\boldsymbol{\varepsilon}$-BRUE path flow pattern if and only if it solves the following nonlinear complementarity problem (NCP), $\forall i \in \mathcal{P}^w$, $\forall w \in \mathcal{W}$:*

$$0 \leqslant f_i^w \perp C_i^w(\mathbf{f}) + \rho_i^w - \pi^w \geqslant 0, \tag{2.3a}$$

$$0 \leqslant \pi^w \perp d^w - \sum_{i \in \mathcal{P}} f_i^w \geqslant 0. \tag{2.3b}$$

where,

$\perp$: the orthogonal sign representing the inner product of two vectors is zero;

ρ_i^w: the actual indifference level, representing deviation of the chosen route's cost from the minimum cost;

π^w: the highest acceptable path cost within the indifference band ε^w for OD pair w and computed as $\pi^w = \min_{j \in \mathcal{P}^w}\{C_j^w(\mathbf{f}) + \varepsilon^w\}$. Denote a vector of the highest acceptable path costs as $\boldsymbol{\pi} = \{\pi^w\}^{w \in \mathcal{W}}$.

By substituting a vector ρ into the BRUE-NCP, one BRUE path flow pattern can be obtained. The BRUE-NCP formulation provides an approach of solving a BRUE solution.

Note.

1. The BRUE NCP formulation holds for general nonlinear path cost functions. If the affine path cost functional form is employed, Eq. (2.3b) becomes $0 \leqslant \mathbf{f} \perp Q\mathbf{f} + (q + \boldsymbol{\rho}) - \boldsymbol{\pi} \geqslant 0$;
2. The BRUE NCP formulation holds for both fixed and elastic traffic demands. For the fixed travel demand case, Eq. (2.3b) is reduced to $\pi^w \geqslant 0$.

Define a new variable as $\mathbf{z} = \binom{\mathbf{f}}{\boldsymbol{\pi}}$ and $F(\mathbf{z}) = \binom{C(\mathbf{f}) + \boldsymbol{\rho} - \boldsymbol{\pi}}{\mathbf{0}}$, then one BRUE solution associated with an indifference level $\mathbf{0} \leqslant \boldsymbol{\rho} \leqslant \boldsymbol{\varepsilon}$ under fixed travel demands can be reformulated as:

$$\mathcal{F}_{BRUE}^{\varepsilon}(\boldsymbol{\rho}) : 0 \leqslant \mathbf{z} \perp F(\mathbf{z}) \geqslant 0. \tag{2.4}$$

Assume affine path costs $C(\mathbf{f}) = Q\mathbf{f} + q$ and fixed travel demands. Define $F(\mathbf{z}) = \begin{bmatrix} Q\mathbf{f} + q + \boldsymbol{\rho} - \boldsymbol{\pi} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} Q & -1 \\ \mathbf{0} & 0 \end{bmatrix} \binom{\mathbf{f}}{\boldsymbol{\pi}} + \binom{q + \boldsymbol{\rho}}{\mathbf{0}} \triangleq \tilde{Q}\mathbf{z} + \tilde{q}$, where $\tilde{Q} = \begin{bmatrix} Q & -1 \\ \mathbf{0} & 0 \end{bmatrix}$, $\tilde{q} = \binom{q + \boldsymbol{\rho}}{\mathbf{0}}$. Then one BRUE solution associated with an indifference level $0 \leq \boldsymbol{\rho} \leq \boldsymbol{\varepsilon}$ can be solved as follows:

$$\mathcal{F}_{BRUE}^{\varepsilon}(\boldsymbol{\rho}) : 0 \leqslant \mathbf{z} \perp \tilde{Q}\mathbf{z} + \tilde{q} \geqslant 0, \tag{2.5}$$

which is a linear complementarity problem (LCP).

In literature on LCP, the notation $LCP(\tilde{q})$ is adopted by Cottle et al. (2009) to denote one solution to the linear complementarity problem formulated in Eq. (2.5). Here the LCP solution depends on $\tilde{q}$ which is further depends on $\boldsymbol{\rho}$.

2.2. ε-BRUE set construction

For any $\boldsymbol{\rho}$ such that $\mathbf{0} \leqslant \boldsymbol{\rho} \leqslant \boldsymbol{\varepsilon}$, $\forall w \in \mathcal{W}, i \in \mathcal{P}^w$, a BRUE path flow pattern can be treated as a UE under the path cost $C^{\boldsymbol{\rho}}(\mathbf{f}) = Q\mathbf{f} + (q + \boldsymbol{\rho})$ with a constant cost vector $\tilde{q} = q + \boldsymbol{\rho}$. Therefore, $\mathcal{F}_{BRUE}^{\varepsilon} = \bigcup_{\mathbf{0} \leqslant \boldsymbol{\rho} \leqslant \boldsymbol{\varepsilon}} \mathcal{F}_{UE}^{\boldsymbol{\rho}}$, where $\mathcal{F}_{UE}^{\boldsymbol{\rho}} = \{\mathbf{f} \in \mathcal{F} : 0 \leqslant \mathbf{f} \perp C^{\boldsymbol{\rho}}(\mathbf{f}) - \boldsymbol{\pi} \geqslant 0, \boldsymbol{\pi} \geqslant \mathbf{0}\}$. In other words, BRUE is a collection of UEs with perturbations in path costs ranging $\mathbf{0} \leq \boldsymbol{\rho} \leq \boldsymbol{\varepsilon}$.

The ε-BRUE solution set is shown to be non-convex (Di et al., 2013; Lou et al., 2010) and thus not easy to construct. By studying its topology, Di et al. (2013) proposed that a BRUE solution set can be further decomposed into multiple subsets. Each subset utilizes the different combinations of paths from the feasible path set. Denote the k^{th}, $k = 1, \ldots, K$ subset as $\mathcal{F}_k^{\varepsilon}$ and its associated acceptable path set as $\mathcal{P}_k^{\varepsilon}$. An acceptable path set includes all paths whose path costs are within the minimum cost plus the indifference band. The smallest acceptable path set is the shortest path set under UE, denoted as $\mathcal{P}_{UE} \triangleq \mathcal{P}_0^{\varepsilon}$. Built upon $\mathcal{P}_{UE}$(i.e., $\mathcal{P}_0^{\varepsilon}$), the acceptable path set $\mathcal{P}_k^{\varepsilon}$, $k = 1, \ldots, K$ can be obtained iteratively by solving a sequence of mathematical programs with equilibrium constraints (MPEC). Interested readers can refer to Di et al. (2013) for the detailed procedure of constructing BRUE subsets. With $\mathcal{P}_k^{\varepsilon}$, $k = 0, \ldots, K$, the k^{th} subset is defined as:

$$\begin{aligned} \mathcal{F}_0^{\varepsilon} \triangleq \{\mathbf{f} \in \mathcal{F} : &\ \forall\, i \in \mathcal{P}_{UE} : f_i, f_j \geqslant 0, |C_i(\mathbf{f}) - C_j(\mathbf{f})| \leqslant \varepsilon; \\ &\ \forall\, i \notin \mathcal{P}_{UE} : f_i = 0\}. \end{aligned} \tag{2.6}$$

$$\begin{aligned} \mathcal{F}_k^{\varepsilon} \triangleq \{\mathbf{f} \in \mathcal{F} : &\ \forall\, i \in \mathcal{P}_{k-1}^{\varepsilon} : f_i \geqslant 0; \exists i \in \mathcal{P}_k^{\varepsilon} \backslash \mathcal{P}_{k-1}^{\varepsilon} : f_i > 0; \\ &\ \forall\, i, j \in \mathcal{P}_k^{\varepsilon} : |C_i(\mathbf{f}) - C_j(\mathbf{f})| \leqslant \varepsilon; \\ &\ \forall\, i \notin \mathcal{P}_k^{\varepsilon} : f_i = 0\}, \quad k = 1, \ldots, K. \end{aligned}$$

where,

$\mathcal{P}_{UE}$: the shortest path set under UE;
$\mathcal{P}_k$: the acceptable path set for subset k;
$\backslash$: set difference operation. For example, $\mathcal{P}_k^\varepsilon \backslash \mathcal{P}_{k-1}^\varepsilon$ denotes all paths which are included in the k^{th} acceptable path set $\mathcal{P}_k^\varepsilon$ but not in $\mathcal{P}_{k-1}^\varepsilon$.

The path flow subset $\mathcal{F}_0^\varepsilon$ corresponding to $\mathcal{P}_{UE}$ contains all feasible path flow patterns where only the UE shortest paths carry flows and the cost difference between any two shortest paths are within the band ε. For the subset $\mathcal{F}_k^\varepsilon$ associated with $\mathcal{P}_k^\varepsilon$, $k = 1, \dots, K$, the newly acceptable paths will carry flows, while those paths belonging to $\mathcal{P}_{k-1}^\varepsilon$ can carry flow or not, and the cost difference between any two acceptable paths are within the band ε. These acceptable path subsets are disjoint, i.e., $\bigcap_{k=0}^K \mathcal{F}_k^\varepsilon = \emptyset$; and it is a subset of the BRUE path flow set $\mathcal{F}_{BRUE}^\varepsilon$.

Then the BRUE solution set is the union of K subsets, where K is the total number of subsets:

$$\mathcal{F}_{BRUE}^\varepsilon = \bigcup_{k=1}^{K} \mathcal{F}_k^\varepsilon. \tag{2.7}$$

Note. K actually depends on ε and should be written as $K \triangleq K(\varepsilon)$. But we will omit writing its dependency in the rest of the paper for the purpose of simplicity.

2.3. A tolled BRUE flow set

When tolls are charged in a road network, the 'tolled BRUE flow set' is defined as $\mathcal{F}_{BRUE}^\varepsilon(\mathbf{y})$, where $\mathbf{y} \geq \mathbf{0}$ is the non-negative toll vector. Similar to Eq. (2.7),

$$\mathcal{F}_{BRUE}^\varepsilon(\mathbf{y}) = \bigcup_{k=1}^{K} \mathcal{F}_k^\varepsilon(\mathbf{y}), \tag{2.8}$$

where the k^{th} tolled subset $\mathcal{F}_k^\varepsilon(\mathbf{y})$ is the solution of the following inequalities:

$$\begin{aligned} \mathcal{F}_0^\varepsilon(\mathbf{y}) \triangleq \{\mathbf{f} \in \mathcal{F} : & \ \forall\, i \in \mathcal{P}_{UE}(\mathbf{y}) : f_i, f_j \geqslant 0, |C_i(\mathbf{f}) + (\mathbf{y}_p)_i - C_j(\mathbf{f}) - (\mathbf{y}_p)_j| \leqslant \varepsilon; \\ & \ \forall\, i \notin \mathcal{P}_{UE}(\mathbf{y}) : f_i = 0\}. \end{aligned} \tag{2.9}$$

$$\begin{aligned} \mathcal{F}_k^\varepsilon(\mathbf{y}) \triangleq \{\mathbf{f} \in \mathcal{F} : & \ \forall\, i \in \mathcal{P}_{k-1}^\varepsilon(\mathbf{y}) : f_i \geqslant 0; \exists i \in \mathcal{P}_k^\varepsilon(\mathbf{y}) \backslash \mathcal{P}_{k-1}^\varepsilon(\mathbf{y}) : f_i > 0; \\ & \ \forall\, i, j \in \mathcal{P}_k^\varepsilon(\mathbf{y}) : |C_i(\mathbf{f}) + (\mathbf{y}_p)_i - C_j(\mathbf{f}) - (\mathbf{y}_p)_j| \leqslant \varepsilon; \\ & \ \forall\, i \notin \mathcal{P}_k^\varepsilon(\mathbf{y}) : f_i = 0\}, k = 1, \dots, K. \end{aligned}$$

where $\mathbf{y}_p$ is the path toll vector. The subscript p will be omitted in the rest of the paper for the purpose of simplicity.

Note.

1. The path flow depends on the toll, i.e., $\mathbf{f} \triangleq \mathbf{f}(\mathbf{y})$. But we will omit its argument for the purpose of simplicity.
2. The total number of tolled BRUE subsets depends on not only ε but also $\mathbf{y}$, i.e., $K \triangleq K(\varepsilon, \mathbf{y})$. But we will omit writing its dependency in the rest of the paper for the purpose of simplicity.

Built upon Eq. (2.5) given affine cost functions, one tolled BRUE flow pattern associated with an indifference level $\mathbf{0} \leq \boldsymbol{\rho} \leq \boldsymbol{\varepsilon}$ can be formulated as a LCP as well, where $F(\mathbf{z}) = \left[\begin{smallmatrix} Q\mathbf{f}+q+\boldsymbol{\rho}+\mathbf{y}-\boldsymbol{\pi} \\ \mathbf{0} \end{smallmatrix}\right] \triangleq \tilde{Q}\mathbf{z} + \tilde{q}$, where $\tilde{Q} = \left[\begin{smallmatrix} Q & -1 \\ \mathbf{0} & 0 \end{smallmatrix}\right]$, $\tilde{q} = \left(\begin{smallmatrix} q+\boldsymbol{\rho}+\mathbf{y} \\ \mathbf{0} \end{smallmatrix}\right)$. The toll $\mathbf{y}$ can be also treated as a perturbation in $\tilde{q}$.

3. Topological properties of the tolled BRUE set

Given a toll vector $\mathbf{y} \geq \mathbf{0}$, $\mathcal{F}_{BRUE}^\varepsilon(\mathbf{y})$ is generally non-unique. The tolled BRUE set can be treated as a point-to-set mapping with respect to the toll vector.

Definition 3.1 (Luderer et al., 2002)**.** Let $\mathcal{Y} = \mathbb{R}^n$, $\mathcal{Z} = \mathbb{R}^m$. A mapping $F : \mathcal{Y} \to 2^{\mathcal{Z}}$ is called *point-to-set mapping* or *multivalued mapping* if every point $y \in \mathcal{Y}$ is associated with a set $F(y) \subset \mathcal{Z}$, where $2^{\mathcal{Z}}$ represents the power set of $\mathcal{Z}$, which is the set of all subsets of $\mathcal{Z}$, including the empty set and $\mathcal{Z}$ itself.

Note. Though the notation $\mathcal{Y}$ represents the feasible toll set in Definition (2.1), here we abuse it a bit to indicate a generic set. We will show in the following that the mapping from the toll set to the BRUE path flow set is actually a point-to-set mapping.

We will explore the topological properties of the point-to-set mapping in the tolled BRUE set. The topological properties of the point-to-set mapping include compactness and upper semi-continuity. Compactness is critical for developing efficient algorithms of solving the tolled BRUE set, while semi-continuity ensures the attainment of optimal solutions in BR-NDP.

3.1. Compactness

Heine–Borel theorem (Mendelson, 1975) says that, if a set is closed and bounded, then it is compact. So boundedness and closedness will be established first.

Proposition 3.1. *$\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}), \forall \mathbf{y} \in \mathcal{Y}$ is bounded.*

Proof. Since $\mathcal{F}$ is bounded, and $\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}) \subset \mathcal{F}$. Thus $\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})$ is bounded. □

The definition of closedness of a point-to-set mapping is given as follows:

Definition 3.2 (Cottle et al., 2009)**.** The solution map $LCP(q)$ is closed if $\{q^{\nu}\}$ is a sequence of vectors in $\mathbb{R}^n$ converging to $\bar{q}$, and $\{z^{\nu}\}$ is a corresponding sequence of solutions with $z^{\nu} \in LCP(q^{\nu})$ for all ν, and if $\{z^{\nu}\}$ converges to $\bar{z}$, then $\bar{z}$ is a solution of $LCP(\bar{q})$.

Using the definition of closedness for the BRUE subset is not straightforward. Instead, we can take advantage of the fact that a BRUE set can be decomposed to multiple subsets defined in Eq. (2.9).

Proposition 3.2. *$\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}), \forall \mathbf{y} \in \mathcal{Y}$ is closed.*

Proof. For each $k \in \{1, \ldots, K\}, \mathcal{F}^{\varepsilon}_{k}(\mathbf{y})$ defined by a system of linear inequalities (2.9) over $\mathcal{Y}$ is obviously closed. Since the union of the closed sets are closed, so $\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})$ is closed. One thing worth of note is that $\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}) = \emptyset, y \notin \mathcal{K}_{\mathcal{Y}}$ but this is trivial and $\emptyset$ is still closed. □

Following the two propositions above and Heine–Borel theorem, we have:

Corollary 1. *$\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}), \forall \mathbf{y} \in \mathcal{Y}$ is compact.*

Actually $\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})$ is not only compact, it is uniformly bounded (Luderer et al., 2002) or uniformly compact (Hogan, 1973).

Definition 3.3 (Luderer et al., 2002)**.** A multivalued mapping F is *uniformly bounded* at $y_0 \in \mathcal{Y}$ if there exists a compact set $Z_0 \subset \mathcal{Z}$ and a neighborhood $\mathcal{Y}_0$ of y_0 such that $F(\mathcal{Y}_0) \subset \mathcal{Z}_0$.

Corollary 2. *$\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}), \forall \mathbf{y} \in \mathcal{Y}$ is uniformly bounded.*

Proof. According to Definition (3.3) of uniformly bounded, set $\mathcal{Z}_0 = \mathcal{F}$ where $\mathcal{F}$ is the feasible path flow set. As $\mathcal{F}$ is compact, for any $y \in \mathcal{Y}, \mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}) \subset \mathcal{F}$ is uniformly bounded. □

3.2. Convexity

To show the BRUE set is a union of convex subsets is equivalent to show that it is polyhedral. According to Cottle et al. (2009), $LCP(q)$ defined in Eq. (2.5) is polyhedral.

Definition 3.4. The solution map $LCP(q)$ is polyhedral if its graph

$$\left\{(q, z) \in \mathbb{R}^n \times \mathbb{R}^n : z \in LCP(q)\right\} \tag{3.1}$$

is a finite union of convex polyhedra.

Define $\mathbf{z} = \binom{\mathbf{f}}{\boldsymbol{\pi}}$ and $\tilde{q} = \binom{q+\boldsymbol{\rho}+\mathbf{y}}{\mathbf{0}}$. Then based on definition of being polyhedral, we have that $\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})$ is polyhedral, meaning it is a finite union of convex polyhedra. The methodology of constructing BRUE sets shown in Eq. (2.8) reflects its polyhedral topology. Eq. (2.9) illustrates how to characterize each convex polyhedra.

We can obtain an immediate result from Inequalities (2.6) that $\mathcal{F}^{\varepsilon}_{k}$ is a bounded intersection of a finite set of half-spaces and thus a convex polytope. However, the union of convex sets are not necessarily convex; furthermore, the number of acceptable paths each polyhedron contains is different, so if multiple subsets are included, the union of polytopes in different dimensions is surely not convex. In summary, given a toll vector $\mathbf{y} \in \mathcal{Y}$ and affine cost functions, $\mathcal{F}^{\varepsilon}_{k}(\mathbf{y})$ is a convex polytope (i.e., a bounded intersection of a finite set of half-spaces); however, $\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})$ may be non-convex if it contains multiple subsets. Similar results can be found in Lou et al. (2010).

The property of the untolled k^{th} subset, i.e., $\mathcal{F}^{\varepsilon}_{k}(\mathbf{0})$, is summarized as follows:

Theorem 3.3. *If the link cost function t is continuous and affine with respect to the link flow vector $\mathbf{x}$, i.e., $t(\mathbf{x}) = A\mathbf{x} + b$, where A is positive-definite. When $\mathcal{F}$ is nonempty, compact and convex, then $\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{0})$ has a nonempty, compact, and generally nonconvex solution set, while each subset $\mathcal{F}^{\varepsilon}_{k}(\mathbf{0})$ defined in Eq. (2.6) has a nonempty, compact, and convex solution set.*

The above analysis of the topological properties focuses on the untolled BRUE set. We can also get a straightforward implication from Theorem (3.3) for the tolled BRUE set:

Corollary 3. *If the link cost function t is continuous and affine with respect to the link flow vector $\mathbf{x}$, i.e., $t(\mathbf{x}) = A\mathbf{x} + b$, where A is positive-definite. When $\mathcal{F}$ is nonempty, compact, and convex, given a toll $\mathbf{y}$, $\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})$ has a nonempty, compact and generally nonconvex solution set for any given and finite vector $\mathbf{y}$, while each subset $\mathcal{F}^{\varepsilon}_{k}(\mathbf{y}), \forall \mathbf{y} \in \mathcal{Y}$ defined in Eq. (2.9) has a nonempty, compact, and convex solution set.*

3.3. Upper-semicontinuity

Definition 3.5 (Luderer et al., 2002)**.** The mapping F is *upper semicontinuous in the sense of Hausdorff* at a point $y_0 \in \mathcal{Y}$if $\forall \epsilon > 0$ there exists a neighborhood W of y_0 such that

$$F(y) \subset F(y_0) + \epsilon B, \forall y \in W. \tag{3.2}$$

where B is an open unit ball with center at 0.

Theorem 3.4 (Theorem 7.2.1 in Cottle et al. (2009))**.** *Let $Q \in \mathbb{R}^{n\times n}$and $\bar{q} \in \mathbb{R}^n$be given. Then there exist a constant $c > 0$ and a neighborhood $V \subseteq \mathbb{R}^n$of $\bar{q}$such that for all vectors $q \in V$,*

$$LCP(q) \subseteq LCP(\bar{q}) + c\|q - \bar{q}\|B. \tag{3.3}$$

In other words, $LCP(q)$ is upper semicontinuous (Luderer et al., 2002) or upper Lipschitzian (Cottle et al., 2009). Upper-semicontinuity states that the solutions of the perturbed $LCP(q)$ not only are not too far away from some solutions of $LCP(q_0)$, but their distances are bounded by a quantity proportional to the magnitude of the change in q_0.

According to the above theorem, we have the following conclusion for the mapping $\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})$:

Proposition 3.5. *$\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}), \forall \varepsilon \geqslant 0, \forall \mathbf{y} \in \mathcal{Y}$is upper semicontinuous.*

Proof. Denote $\mathbf{X} = \binom{\mathbf{y}}{\boldsymbol{\rho}}, \mathbf{X_0} = \binom{\mathbf{y}_0}{\boldsymbol{\rho}_0}, 0 \leqslant \boldsymbol{\rho} \leqslant \varepsilon$. One BRUE solution associated with the indifference function $\boldsymbol{\rho}$ is denoted as $\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}, \boldsymbol{\rho})$,then $\exists c > 0$,

$$\begin{aligned}
\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}, \boldsymbol{\rho}) &\triangleq \mathcal{F}^{\varepsilon}_{BRUE}(X) \subseteq \mathcal{F}^{\varepsilon}_{BRUE}(X_0) + c\|X - X_0\|B \\
&= \mathcal{F}^{\varepsilon}_{BRUE}(X_0) + c\left\| \begin{pmatrix} y \\ \boldsymbol{\rho} \end{pmatrix} - \begin{pmatrix} \mathbf{y}_0 \\ \boldsymbol{\rho}_0 \end{pmatrix} \right\| B \\
&\subseteq \mathcal{F}^{\varepsilon}_{BRUE}(X_0) + c\|\mathbf{y} - \mathbf{y}_0\|B + c\|\boldsymbol{\rho} - \rho_0\|B \\
&\subseteq \mathcal{F}^{\varepsilon}_{BRUE}(X_0) + c\|\mathbf{y} - \mathbf{y}_0\|B + c\varepsilon B \\
&\subseteq \mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}_0, \boldsymbol{\rho}_0) + c(\|\mathbf{y} - \mathbf{y}_0\| + \varepsilon)\|B.
\end{aligned}$$

Let $M_y \triangleq \max\|\mathbf{y} - \mathbf{y}_0\|, \epsilon = c(M_y + \varepsilon)$,then $\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}) \subseteq \mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}_0) + \epsilon B$. In other words, $\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}), \forall \varepsilon \geqslant 0$is upper semicontinuous with respect to $\mathbf{y}$. □

Denote the feasible toll which ensures the existence of the k^{th} BRUE subset as:

$$\mathcal{K}^k_y = \left\{y : \mathcal{F}^{\varepsilon}_k(\mathbf{y}) \neq \emptyset\right\} \cap \mathcal{Y}, \tag{3.4}$$

which implies that, $\mathbf{y} \in \mathcal{K}^k_y \Rightarrow \exists \mathbf{f} \in \mathcal{F}^{\varepsilon}_k(\mathbf{y}) : f_i \geqslant 0, i \in \mathcal{P}^{\varepsilon}_k; f_j = 0, j \notin \mathcal{P}^{\varepsilon}_k$,where $\mathcal{P}^{\varepsilon}_k$is the acceptable path set for the k^{th} subset. Then the feasible toll which ensures the existence of the BRUE set is:

$$\mathcal{K}_y = \{y \in \mathcal{Y} : \mathcal{F}^{\boldsymbol{\varepsilon}}_{BRUE} \neq \emptyset\} = \bigcup_{k=1}^{K} \mathcal{K}^k_y. \tag{3.5}$$

In the following, we will show that $\mathcal{K}_y$is a compact set.

Corollary 4 (Hogan, 1973)**.** *If F is upper semicontinuous on $\mathcal{Y}$and uniformly compact at $y \in \mathcal{Y}$,then the set $\mathcal{K}_y = \left\{y \in \mathcal{Y} : F(y) \neq \emptyset\right\}$is closed in $\mathcal{Y}$.*

Combining the results from Corollary (2) and Proposition (3.5) along with Corollary (4), now we can show automatically that:

Corollary 5. *$\mathcal{K}_y$defined in Eq. (3.5) is compact.*

The compactness of the set $\mathcal{K}_y$will help prove the solution existence of BR toll pricing.

4. Toll pricing under bounded rationality (BR-TP)

4.1. Risk-averse, risk-prone, and risk-neutral toll pricing

Non-uniqueness of BRUE flow patterns generates risks in planning. Urban planners may hold different attitudes towards toll pricing:

- If urban planners assume the worst case is most likely to happen when implementing toll pricing and the best case is very possible to occur without it, then we say that this planner has an attitude preference of 'risk-averse' towards toll pricing.
- If urban planners assume the best case is most likely to happen when implementing toll pricing while the worst case occurs most likely without it, then this planner holds an attitude of 'risk-prone' towards toll pricing.

- If the planner believes each BRUE has a certain probability to occur when implementing toll pricing and only expected BRUE is considered, we call this attitude as 'risk-neutral'.

In the following, we would like to explore the impact of different attitudes towards toll pricing on planning decisions. Denote STT as the total system travel time given a toll vector $\mathbf{y}$, then $STT(\mathbf{y},\mathbf{f}) \triangleq \sum_{i\in\mathcal{P}^w} f_i C_i(y_i, f_i)$. As STT is a continuous function with respect to path flows and the BRUE set is compact, the maximum and the minimum values of STT over the BRUE flow set can be attained and are defined as follows:

Definition 4.1. For a given traffic demand $d \geq 0$ and an indifference band $\varepsilon \geq 0$, the worst, the best and the expected system travel time (denoted as 'w-STT', 'b-STT' and 'ESTT'respectively) are defined as the maximum, the minimum and the mean of STT among the whole $\boldsymbol{\varepsilon}$-BRUE solution set, i.e.,

$$\text{w-}STT(\mathbf{y},\mathbf{f}) = \max_{\mathbf{f}\in\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})} \sum_{i\in\mathcal{P}^w} f_i C_p(y_i, f_i), \tag{4.1a}$$

$$\text{b-STT}(\mathbf{y},\mathbf{f}) = \min_{\mathbf{f}\in\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})} \sum_{i\in\mathcal{P}^w} f_i C_p(y_i, f_i). \tag{4.1b}$$

$$ESTT = \int_{\text{b-}STT}^{\text{w-}STT} g(STT)dSTT. \tag{4.1c}$$

where $g(STT)$ is the probability distribution function over the STT interval and $\int_{\text{b-}STT}^{\text{w-}STT} STT dSTT = 1$.

Based on different attitudes towards risks caused by non-uniqueness of BRUE, the following scenarios for toll pricing under bounded rationality (BR-TP) can be defined:

Definition 4.2.

(a) Risk-averse BR-TP (denoted as 'BR-TP-RA') is to find an optimal toll vector such that the worst STT is minimized;
(b) Risk-prone BR-TP (denoted as 'BR-TP-RP') is to find an optimal toll vector such that the best STT is minimized;
(c) Risk-neutral BR-TP (denoted as 'BR-TP-RN') is to find an optimal toll vector such that the expected STT is minimized given a probability distribution over the BRUE set;
(d) BR-TP under UE (denoted as 'BR-TP-UE') is to find an optimal toll vector such that the STT under UE is minimized.

In this study, we mainly focus on BR-TP-RA and BR-TP-RP.

4.2. Risk-averse and risk-prone BR-TP formulation

BR-TP-RP and BR-TP-RA are to minimize the total system cost in the best and the worst case under the BR behavioral assumption, by deciding optimal tolls charged on the candidate link set. So they can be formulated as:

$$\begin{aligned} &\min_{\mathbf{y}\in\mathcal{Y}} \min_{\mathbf{x}\in\mathcal{X}} / \max_{\mathbf{x}\in\mathcal{X}} \sum_{a\in\mathcal{A}} x_a t_a(x_a) + \sum_{a\in\mathcal{T}} x_a t_a(y_a, x_a) \\ &s.t. \qquad 0 \leqslant y_a \leqslant \bar{y}, a \in \mathcal{T} \\ &\qquad\qquad \mathbf{x} = \Delta\mathbf{f}, \forall \mathbf{f} \in \mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}). \end{aligned} \tag{4.2}$$

where, y_a: the toll charged on link a, between zero and an upper bound $\bar{y}$;
x_a: the link flow for link a;
$\mathcal{A}$: the link set without tolls charged;
$\mathcal{T}$: the link set with tolls charged;
$\mathbf{x} = \{x_a\}_{a\in\mathcal{L}}$: the link flow vector;
$t_a(y_a, x_a)$: the cost of traversing link a, which depends on total traffic using link a and the toll charged on link a;
$\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})$: the tolled BRUE solution set defined by Eq. (2.9).
The link-based BR-TP can be rewritten as the path-based formulation:

$$\begin{aligned} &\min_{\mathbf{y}\in\mathcal{K}_y} \min_{\mathbf{f}\in\mathcal{F}^{\varepsilon}_{\mathbf{BRUE}}(\mathbf{y})} / \max_{\mathbf{f}\in\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})} \sum_{i\in\mathcal{P}^w} f_i C_i(y_i, f_i) \\ &s.t. \qquad 0 \leqslant y_i \leqslant n_p \bar{y}, i \in \mathcal{P}^w \\ &\qquad\qquad \mathbf{f} \in \mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y}). \end{aligned} \tag{4.3}$$

where n_p is the number of links along the path p.
Recall that we define $STT(\mathbf{y},\mathbf{f}) = \sum_{i\in\mathcal{P}^w} f_i C_i(y_i, f_i)$. Then BR-TP can be rewritten compactly as:

$$\min_{\mathbf{y}\in\mathcal{K}_y} \min_{\mathbf{f}\in\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})} / \max_{\mathbf{f}\in\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})} STT(\mathbf{y},\mathbf{f}). \tag{4.4}$$

5. Solution existence conditions

5.1. Risk-prone BR-TP

Solution existence of BR-TP-RP can be proven by decomposing BR-TP-RP into multiple subproblems defined over BRUE subsets.

Let

$$\Psi_k(\mathbf{y}) \triangleq \begin{cases} \min_{\mathbf{f}\in\mathcal{F}_k^\varepsilon(\mathbf{y})} STT(\mathbf{y},\mathbf{f}), y \in \mathcal{K}_y, \\ M, \text{ o.w.} \end{cases}, \tag{5.1}$$

where M is a very big number to ensure that $\Psi(\mathbf{y}) \neq \Psi_k(\mathbf{y})$ if $\mathcal{F}_k^\varepsilon(\mathbf{y}) = \emptyset$(in other words, $y \notin \mathcal{K}_y$).

Then BR-TP-RP can be rewritten as: $\Psi(\mathbf{y}) = \min_{k=\{0,\dots,K\}} \Psi_k(\mathbf{y}), \forall \mathbf{y} \in \mathcal{Y}$. Note that $\Psi(\mathbf{y})$ is also defined over the interval where the k^{th}subset does not exist, so it is defined over the feasible toll set $\mathbf{y} \in \mathcal{Y}$.

Solving BR-TP-RP is then transformed to obtaining the minima for $K+1$subproblems $\Psi_k(\mathbf{y})$, and then find the minima among these $K+1$optima. $\Psi_k(\mathbf{y})$ can be reformulated as:

$$\Psi_k(\mathbf{y}) : \min_{\mathbf{y},\mathbf{f}(\mathbf{y})} STT(\mathbf{y},\mathbf{f}(\mathbf{y})) \tag{5.2a}$$

$$s.t.(\mathbf{y},\mathbf{f}(\mathbf{y})) \in G. \tag{5.2b}$$

where,

$$G = \left\{(\mathbf{y},\mathbf{f}(\mathbf{y}))|\mathbf{f}(\mathbf{y}) \in \mathcal{F}_k^\varepsilon(\mathbf{y}), \mathbf{y} \in \mathcal{K}_y^k\right\}. \tag{5.3}$$

Before showing its solution existence, we need to first introduce a theorem.

Theorem 5.1 (Theorem 1 in Ban et al. (2009))**.** *If the link cost function t is continuous and monotone with respect to the link flow vector* $\mathbf{x}$, *STT*($\mathbf{y}, \mathbf{f}$) *is continuous with respect to* ($\mathbf{y}, \mathbf{f}$), *and* $\mathcal{F}$*is nonempty, compact, and convex, then the following two statements hold:*

- *G is compact;*
- $\Psi_k(\mathbf{y})$ *has at least one solution.*

Based on Theorem (5.1), we know that there exists optimal solutions for $\Psi_k(\mathbf{y})$ in BR-TP-RP. Then there must also exist optimal solutions for the minimum over K finite solutions, i.e., there exists at least one solution for $\Psi(\mathbf{y}) = \min_{k=1}^{K} \Psi_k(\mathbf{y})$. In other words, BR-TP-RP has optimal solutions and the solution existence condition is established. Note that this solution may be non-unique.

5.2. Risk-averse BR-TP

Theorem 5.2 (Ban et al., 2009)**.** *If the link cost function t is continuous and monotone with respect to the link flow vector* $\mathbf{x}$, *STT*($\mathbf{y}, \mathbf{f}$) *is continuous with respect to* $\mathbf{y}$ *and* $\mathbf{f}(\mathbf{y})$, *and* $\mathcal{F}$*is nonempty, compact, and convex. then,*

$$\max_{\mathbf{f}\in\mathcal{F}_k^\varepsilon(\mathbf{y})} STT(\mathbf{y},\mathbf{f}) > -\infty, \tag{5.4}$$

Define

$$\Phi_k(\mathbf{y}) \triangleq \begin{cases} \max_{\mathbf{f}\in\mathcal{F}_k^\varepsilon(\mathbf{y})} STT(\mathbf{y},\mathbf{f}), y \in \mathcal{K}_y^k, \\ 0, \text{ o.w.} \end{cases} \tag{5.5}$$

Then we have

$$\Phi(\mathbf{y}) \triangleq \max_{\mathbf{f}\in\mathcal{F}_k^\varepsilon(\mathbf{y})} STT(\mathbf{y},\mathbf{f}), \forall \mathbf{y} \in \mathcal{Y}. \tag{5.6}$$

Note again that Eq. (5.5) is also defined over the interval where the k^{th} subset does not exist, so it is defined over the feasible toll set $\mathbf{y} \in \mathcal{Y}$.

The above theorem indicates that $\Phi_k(\mathbf{y})$ has a greatest lower bound as y varies within $\mathcal{Y}$. Since $\mathcal{Y}$is a bounded set, there must also exist optimal solutions for $\Phi_k(\mathbf{y})$ and therefore $\Phi(\mathbf{y})$.

Ban et al. (2009) then gave a theorem of the solution existence condition for BR-TP-RA:

Theorem 5.3. *Suppose that the set* $\mathcal{K}_y^k$*is compact, and the function* $\Phi_k(\mathbf{y})$ *defined in Eq. (5.5) is lower semicontinuous at each point* $\mathbf{y}$ *in* $\mathcal{K}_y^k$*,then the BR-TP-RA has at least one solution.*

To guarantee the attainment of an optimal solution of BR-TP-RA, $\Phi(\mathbf{y})$ needs to be lower semicontinuous. In the following, we will investigate the semi-continuity of $\Phi(\mathbf{y})$.

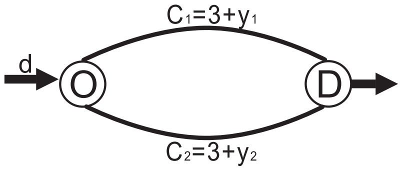


Fig. 1. A network with two links.

In multi-valued mapping, $\Phi(\mathbf{y})$ is called the supreme 'marginal function' (Luderer et al., 2002). Its continuity property is guaranteed by the following theorem:

Theorem 5.4 (Hogan, 1973; Luderer et al., 2002)**.**

$$\Phi(y) = \sup_{z \in F(y_0)} f(y, z). \tag{5.7}$$

If F is upper semicontinuous at y_0 and uniformly compact near y_0, and if f is upper semicontinuous on $y_0 \times F(y_0)$, then $\Phi(y)$ is upper semicontinuous at x_0.

Consequently, the above theorem can be applied to show the upper-semicontinuity of $\Phi(\mathbf{y})$:

Proposition 5.5. $\Phi(\mathbf{y}), \mathbf{y} \in \mathcal{Y}$*defined in Eq. (5.6) is upper semicontinuous at* $\mathbf{y}$.

Proof. By Proposition (3.5), $\mathcal{F}_{BRUE}(\mathbf{y})$is upper-semicontinuous. The system travel time function *STT* is continuous on $\mathbf{y} \times \mathcal{F}_{BRUE}(\mathbf{y})$. Therefore $\Phi(\mathbf{y})$ is upper semicontinuous at $\mathbf{y}$. $\square$

$\Phi(\mathbf{y})$ is upper semicontinuous at each point $\mathbf{y}$ in $\mathcal{Y}$,but its lower-continuity cannot be proven, thus BR-TP-RA may not attain an optimal solution.

Now we will give a simple example to show that $\Phi(\mathbf{y})$ is upper semicontinuous.

Example 5.1. The OD demand is connected by two paths between nodes *O* and *D* illustrated in Fig. 1. Two paths have the same travel cost of 3. Assume tolls y_1 and y_2 are charged on link #1, #2respectively. The demand level is *d* and the indifference band is ε.

There are three cases of constructing the BRUE set:

1. $y_1 - y_2 < -\varepsilon \Rightarrow C_2 - C_1 > \varepsilon$: $f_{BRUE} = [d \;\; 0]^T, \Phi(y_1, y_2) = (y_1 + 3)d$.
2. $-\varepsilon \leqslant y_1 - y_2 \leqslant \varepsilon \Rightarrow -\varepsilon \leqslant C_1 - C_2 \leqslant \varepsilon$: $\mathcal{F}^{\varepsilon}_{BRUE} = \{\mathbf{f} : f_1 + f_2 = d, f_1, f_2 \geqslant 0\}, STT(\mathbf{f}, y) = f_1C_1 + f_2C_2 = (y_1 - y_2)f_1 + d(y_2 + 3)$. Thus $\Phi(y_1, y_2) = \begin{cases} (y_2 + 3)d \text{ if } y_1 - y_2 \leqslant 0, \\ (3 + y_1)d \text{ if } y_1 - y_2 > 0. \end{cases}$.
3. $y_1 - y_2 > \varepsilon \Rightarrow C_1 - C_2 > \varepsilon$: $f_{BRUE} = [0 \;\; d]^T, \Phi(y_1, y_2) = (y_2 + 3)d$.

In summary, $\Phi(y_1, y_2) = \begin{cases} (y_1 + 3)d, & \text{if } y_1 - y_2 < -\varepsilon, \\ (y_2 + 3)d, & \text{if } -\varepsilon \leqslant y_1 - y_2 \leqslant 0, \\ (y_1 + 3)d, & \text{if } 0 < y_1 - y_2 \leqslant \varepsilon, \\ (y_2 + 3)d, & \text{if } y_1 - y_2 > \varepsilon. \end{cases}$ and its plot is shown in Fig. 2.

In Fig. 2, $\Phi(y_1, y_2)$ takes a higher value at discontinuity $y_1 - y_2 = -\varepsilon$or $y_1 - y_2 = \varepsilon$,therefore it is upper semicontinuous.

Remark. Both BR-TP-RP and BR-TP-RA are mathematical programs with equilibrium constraints (MPEC). They are *NP*-hard. Section 5.1 shows that an optimal BR-TP-RP exists theoretically. However, it does not guarantee in practice that the optimal toll can be actually attained using any algorithm. Moreover, Section 5.2 shows that BR-TP-RP may not exist. However, in real-world applications, we should always be able to find a suboptimal toll for the purpose of implementation. Because algorithm is not the main focus of this paper, we will direct readers to Luo et al. (1996) for detailed algorithm implementation of solving MPEC problems.

6. Numerical examples

6.1. An illustrative example of a network with three paths

In the following, we will illustrate how to obtain the optimal toll analytically in the typical Braess network. The topology of the network, the OD demand between nodes $1 - 4$and link cost functions are illustrated in Fig. 3. There are three paths: 1-3-4 (path 1),1-2-4 (path 2) and 1-3-2-4 (path 3). The total demand $d = 6$and the indifference band $\varepsilon = 10$. The toll *y*is charged on link #5.

For the fixed demand level, due to flow conservation, the feasible flow set can be characterized by two path flows, i.e., $f_3 = 6 - f_1 - f_2$. Therefore, three path costs can be computed as functions of flows on the first two paths:

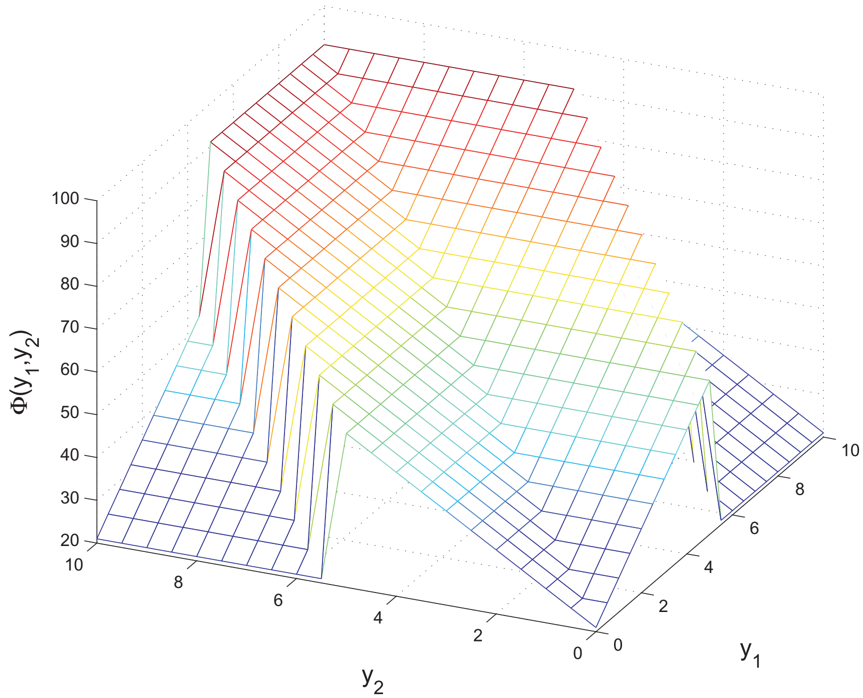


Fig. 2. $\Phi(y_1, y_2)$ given $d = 7, \varepsilon = 5$.

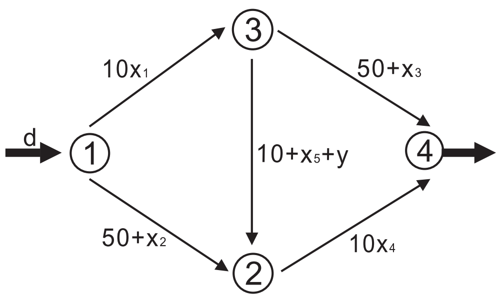


Fig. 3. The network illustration with three paths.

$$
\begin{aligned}
C_1 &= 11f_1 + 10f_3 + 50 = f_1 - 10f_2 + 110,\\
C_2 &= 11f_2 + 10f_3 + 50 = -10f_1 + f_2 + 110,\\
C_3 &= 10(f_1 + f_2) + 21f_3 + 10 + y = -11(f_1 + f_2) + 136 + y.
\end{aligned}
$$

UE is a special BRUE. It can be computed by letting three path costs equal, i.e., $C_1 = C_2 = C_3$, and the flow pattern needs to be feasible, i.e., $f_1 + f_2 + f_3 = 6,\ f_1, f_2, f_3 \geqslant 0$. Then we have

$$
\mathbf{f}_{UE}(y) = \begin{cases} \begin{bmatrix} \frac{y+26}{13} & \frac{y+26}{13} & \frac{-2y+26}{13} \end{bmatrix}^T, & \text{if } 0 \leqslant y < 13, \\ \begin{bmatrix} 3 & 3 & 0 \end{bmatrix}^T, & \text{if } y \geqslant 13. \end{cases} \tag{6.1}
$$

When $0 \leq y < 13$, all three paths are utilized under UE. Therefore under BRUE these three paths are all acceptable. According to Eq. (2.6):

$$
\begin{aligned}
\mathcal{F}_{BRUE}^{\varepsilon=10} = \{\mathbf{f} \in \mathcal{F} :\ & f_1 + f_2 + f_3 = 6,\ f_1, f_2, f_3 \geqslant 0,\\
& |f_1 - f_2| \leqslant \tfrac{10}{11};\\
& |12f_1 + f_2 - 26 - y| \leqslant 10;\\
& |f_1 + 12f_2 - 26 - y| \leqslant 10\}.
\end{aligned}
$$

When $y \geq 13$, only the first two paths are utilized and the third path is too costly to use under UE. Let $\varepsilon_1^*(y)$ be the critical indifference band when more paths are included (and detailed definition is referred to Di et al. (2013)). If $\varepsilon_1^*(y) > 10$, the BRUE set is composed of only one subset: $\mathcal{F}_{BRUE}^{\varepsilon=10} = \mathcal{F}_0^{\varepsilon=10}$. If $\varepsilon_1^*(y) < 10$, the BRUE set is the union of two subsets: $\mathcal{F}_{BRUE}^{\varepsilon=10} = \mathcal{F}_0^{\varepsilon=10} \bigcup \mathcal{F}_1^{\varepsilon=10}$, where each subset is defined as follows:

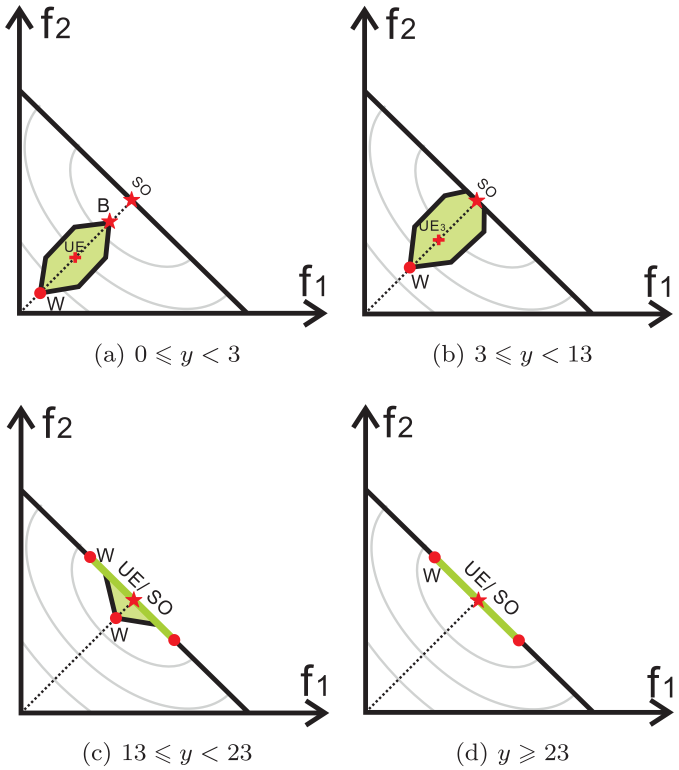


Fig. 4. BRUE set.

$$\mathcal{F}_0^{\varepsilon=10} = \{\mathbf{f} \in \mathcal{F} : f_1 + f_2 = 6, f_1, f_2 \geqslant 0, f_3 = 0, \\ |f_1 - f_2| \leqslant \tfrac{10}{11}\}.$$

$$\mathcal{F}_1^{\varepsilon=10} = \{\mathbf{f} \in \mathcal{F} : f_1 + f_2 + f_3 = 6, f_1, f_2, f_3 \geqslant 0, \\ |f_1 - f_2| \leqslant \tfrac{10}{11}; \\ |12f_1 + f_2 - 26 - y| \leqslant 10; \\ |f_1 + 12f_2 - 26 - y| \leqslant 10\}.$$

In summary, when $13 \le y < 23$, the BRUE is the union of two subsets: $\mathcal{F}_0^{\varepsilon=10}$with first two paths utilized and $\mathcal{F}_1^{\varepsilon=10}$with all three paths utilized (See in Fig. 4c). When $y > 23$, only first two paths can be utilized and the BRUE set is $\mathcal{F}_0^{\varepsilon=10}$.

In Fig. 4a–d, the triangle area encompassed by two axes and the line $f_1 + f_2 = 6$is the feasible region. UE is plotted as a red cross in Fig. 4. The BRUE set is indicated by a green region. To consider the feasibility of the BRUE set and whether the system optimal (SO) (i.e., the feasible flow pattern with the lowest STT values) is included in the BRUE set, we further divide the toll into four intervals, and the BRUE set given each toll range is plotted in Fig. 4a–d.

To find *b*-STT and *w*-STT, we need to inspect the STT function. STT is formulated as:

$$STT(\mathbf{f}, y) = \sum_{i=1}^{3} C_i f_i = 12(f_1^2 + f_2^2) + 2f_1 f_2 + (18 - y)(f_1 + f_2) + (6y + 816). \tag{6.2}$$

Eq. (6.2) shows that given a certain demand level, *STT* is a general conic equation with respect to the path flow pattern **f**. SO is the feasible flow pattern with the lowest STT and can be calculated as: $f_{SO} = [\ 3 \quad 3 \quad 0\ ]^T$, $STT_{SO} = 498$.

In Fig. 4, SO is denoted as a red star, which is the center of STT contour lines (i.e., grey elliptical lines). Each point on a contour line represents the value of *STT* for a certain flow pattern. The contour line which is farther from SO has higher *STT*. Accordingly, the BRUE flow pattern which is located on the farthest STT contour is the worst case; while that on the closest contour is the best case.

To solve risk-averse toll pricing, we need to first identify the worst scenario given y. When $0 \le y < 13$ (shown in Fig. 4a–b), the worst flow pattern is denoted as ‘W’, which is the leftmost corner point. When $13 \le y < 23$ (shown in Fig. 4c), there are two possible worst flow patterns, denoted as ‘W’. When $y \ge 23$ (shown in Fig. 4d), the BRUE set is a line segment along the line $f_1 + f_2 = 6$,and the worst cases are on both ends.

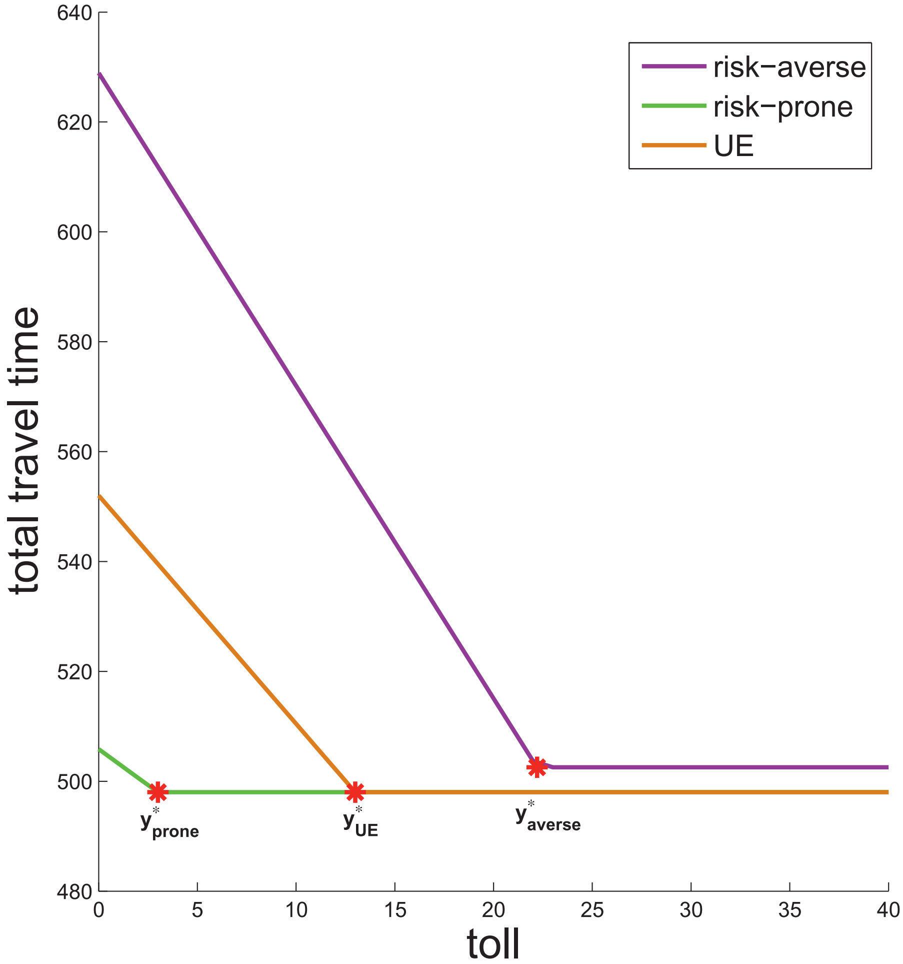


Fig. 5. $\Phi(y)$.

$$\Phi(y)=\begin{cases}\frac{-34y+6576}{13}, & \text{if } 0\leqslant y<13,\\ \max\{\frac{-74y+8176}{13},\frac{5528}{11}\}, & \text{if } 13\leqslant y<23,\\ \frac{5528}{11}, & \text{if } y\geqslant 23.\end{cases} \tag{6.3}$$

To solve risk-prone toll pricing, we need to identify the best scenario given y. When $0\leq y<3$ (shown in Fig. 4a), the best flow pattern is denoted as 'B', which is the rightmost corner point. When $y\geq 3$, SO is included in the BRUE set and thus the best flow pattern is SO (shown in Fig. 4b–d).

$$\Psi(y)=\begin{cases}\frac{-74y+8176}{13}, & \text{if } 0\leqslant y<13,\\ 498, & \text{if } y\geqslant 3.\end{cases} \tag{6.4}$$

Plotting $\Phi(y)$ and $\Psi(y)$ with respect to the toll y (Fig. 5):

The optimal tolls for BR-TP-RP, BR-TP-UE, and BR-TP-RA are ones minimizing respective STT values as y varies. We need to notice that, optimal tolls are not unique: $y^*_{RP}\geqslant 3, y^*_{UE}\geqslant 13,$ and $y^*_{RA}\geqslant 22$. We have the following conclusions:

- The gap between b-STT and w-STT given a specific toll represents variation of the system operational efficiency due to bounded rationality;
- The lower bound of $y^*_{UE}=13$is the marginal cost;
- The lower bound of $y^*_{RP}=3$is smaller than $y^*_{RA}=22$. This indicates that, existence of the indifference band ε can improve the operational efficiency of a transportation network at some flow patterns. When the network operates under these flow patterns, the optimal toll can be charged much less to improve system efficiency;
- With $y^*_{RP}$and y^*_{UE},the system operates at SO with the total system travel time of 498; while with y^*_{RA},the system can run at best with the total system travel time of 503;

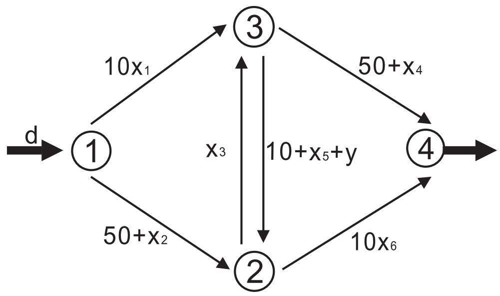


Fig. 6. The network illustration with four paths.

- When the optimal toll y^*_{RP}or $y^*_{UE}$is charged, under the best BRUE flow pattern or the UE, the middle link is not used. In other words, the addition of the middle link causes the Braess paradox to occur and charging the toll will force people not to use it. The effect of the paradox can be removed using toll pricing.

6.2. A numerical example of solving BR-TP-RA

In the second example, we will add one link from node 2 to node 3. The topology of the test network, the OD demand between nodes $1-4$and link cost functions are illustrated in Fig. 6. There are four paths: 1-3-4 (path 1), 1-2-4 (path 4), 1-3-2-4 (path 3), 1-2-3-4 (path 4). The toll y is charged on link 5.

6.2.1. BRUE subsets

In this example, we will decompose the BRUE set into multiple subsets and illustrate the worst scenario $\Phi_k(y), k = 1, \ldots, K,$where K is the number of subsets given y. Thus,

$\Phi(\mathbf{y}) = \max_{k=1}^{K} \Phi_k(\mathbf{y})$.

First, we compute UE with respect to the demand level and the toll:

$$\mathbf{f}_{UE}(y) = \begin{cases} \begin{bmatrix} \frac{11d-40+y}{13} & \frac{11d-40+y}{13} & \frac{-9d+80-2y}{13} & 0 \end{bmatrix}^T, \text{ if } 0 \leqslant d < -\frac{2}{9}y + \frac{80}{9}, \\ \begin{bmatrix} \frac{d}{2} & \frac{d}{2} & 0 & 0 \end{bmatrix}^T, \text{ if } -\frac{2}{9}y + \frac{80}{9} \leqslant d < \frac{100}{9}, \\ \begin{bmatrix} \frac{2d+50}{13} & \frac{2d+50}{13} & 0 & \frac{9d-100}{13} \end{bmatrix}^T, \text{ if } d \geqslant \frac{100}{9}. \end{cases}$$

Assume $d < \frac{100}{9}, -\frac{9}{2}d + 50 \leqslant \varepsilon < -9d + 90$. According to the methodology introduced in Di et al. (2013), we can calculate acceptable path sets for the network shown in Fig. 6 as follows:

$$\mathcal{P}^{\varepsilon} = \begin{cases} \{\{1,2,3\},\{1,2,3,4\}\}, \text{if } 0 \leqslant y < -\frac{9}{2}d + 40, \\ \{\{1,2\},\{1,2,3\},\{1,2,3,4\}\}, \text{if } -\frac{9}{2}d + 40 \leqslant y < -9d + 90, \\ \{\{1,2\},\{1,2,3,4\}\}, \text{if } y \geqslant -9d + 90. \end{cases}$$

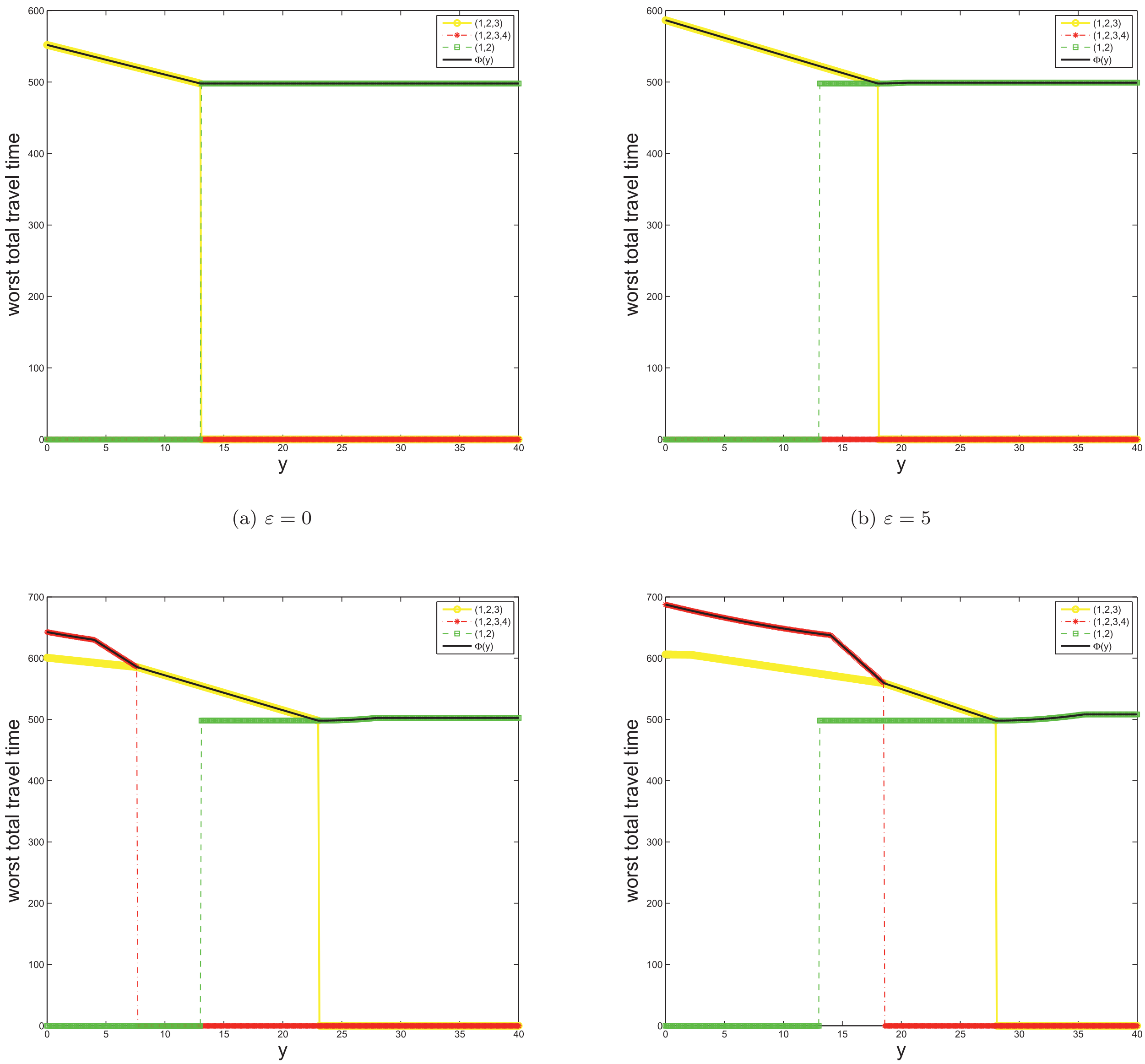
When $0 \leqslant y < -\frac{9}{2}d + 40,$under UE, paths {1, 2, 3} are utilized; path 4 can also be utilized due to the existence of bounded rationality, and therefore {1, 2, 3, 4} is another subset. Given these two acceptable path sets, the BRUE subset is formulated as shown in Eq. (2.6). Then the BRUE set is the union of two subsets with {1, 2, 3} and {1, 2, 3, 4} as acceptable paths respectively. Based on each BRUE subset, we can plot $\Phi_k(y), k = 1, \ldots, K,$where K is the number of subsets given y and may vary as y and ε vary. $\Phi(y)$ is the upper envelop of $\Phi_k(y), k = 1, \ldots, K$.

Given different ε, when the toll varies, the possible subsets vary as well. When $\varepsilon = 0$,i.e., under UE, path 3 will not be used when $y > 13$. When $\varepsilon = 5$,the indifference band is not large enough to allow path 4 to be used. Path 3 is used along with paths 1, 2 when $y \leq 13$ and unused when the toll is large. When $\varepsilon = 10$,paths 3 and 4 are used when the toll is smaller and are not acceptable when the toll is larger. When $\varepsilon = 15$,the path usage shares the similar pattern as when $\varepsilon = 10$.

Take $\varepsilon = 10$for example, we will show how each $\Phi_k(y), k = 0, 1, 2$is calculated. When $0 \leq y \leq 8$, the BRUE set is composed of two subsets: either paths 1, 2, 3 are used or either paths 1, 2, 3, 4 are used. When $8 \leq y < 13$, the BRUE set is composed of only one set with acceptable paths 1, 2, 3. When $13 \leq y \leq 23$, two subsets {1, 2, 3} and {1, 2} exist. When $y > 23$, path 3 will not be used due to high toll charge and only paths 1 and 2 are acceptable. Each subset contains a set of BRUE solutions and the worst flow distribution can be found by solving Eq. (5.5).

As the toll varies, the existence of each subset varies and thus the maximum of each subset also varies. As ε gradually increases, the values of the STT for each subset varies and the shape of $\Phi(y)$ changes accordingly.

Note. One thing we want to point out is that $\Phi(\mathbf{y})$ should always be upper-semicontinuous, meaning that when there is a jump at $\mathbf{y}$, $\Phi(\mathbf{y})$ carries the higher value at jump. In this network, due to its simple topology, $\Phi(y)$ is not only upper-semicontinuous but also lower-semicontinuous, thus continuous. Therefore there exists an optimal toll for this example.



(a) $\varepsilon = 0$

(b) $\varepsilon = 5$

(c) $\varepsilon = 10$

(d) $\varepsilon = 15$

Fig. 7. $\Phi(y)$ given $d = 6$.

In conclusion, the point-to-set mapping $\Phi(y)$ is complicated and the only property we know is upper-semicontinuity, which cannot ensure the attainment of an optimal toll. However, in practice, we can always find a suboptimal toll for BR-TP-RA.

6.2.2. Algorithm of solving an optimal BR-TP-RA

Given $\Phi(\mathbf{y})$ defined in Eq. (5.6), BR-TP-RA can be written as: $\min_{\mathbf{y}\in\mathcal{Y}} \Phi(\mathbf{y})$. Solving BR-TP-RA includes two main steps: given a toll $\mathbf{y}$, solving $\Phi(\mathbf{y})$; and employing some algorithm to find an optimal toll minimizing the total system travel time over all feasible tolls. These two steps will iterate till an optimal solution or a suboptimal solution satisfying a per-defined tolerance range is found. The steps of solving an optimal BR-TP-RA in small networks are listed as follows:

1. Set the n^{th} iteration $n = 0$. The procedure starts with an initial toll $\mathbf{y}^{(0)}$. Given $\mathbf{y}^{(0)}$, evaluating $\Phi(\mathbf{y}^{(0)})$ can be converted to solving multiple subproblems over BRUE subsets. First, UE can be solved and thus the shortest path set $\mathcal{P}_{UE}(\mathbf{y}^{(0)})$is obtained. After knowing $\mathcal{P}_{UE}(\mathbf{y}^{(0)})$,repeat the procedure of solving the acceptable path sets proposed by Di et al. (2013) and we will obtain $K+1$BRUE subsets $\mathcal{F}_k^{\varepsilon}(\mathbf{y}^{(0)})$. Given the toll vector $\mathbf{y}^{(0)}$and $\mathcal{F}_k^{\varepsilon}(\mathbf{y}^{(0)})$,evaluating $\Phi_k(\mathbf{y}^{(0)})$ is to solve a nonlinear complementarity program defined in Eq. (5.5). When affine link cost functions are employed, *STT* function is

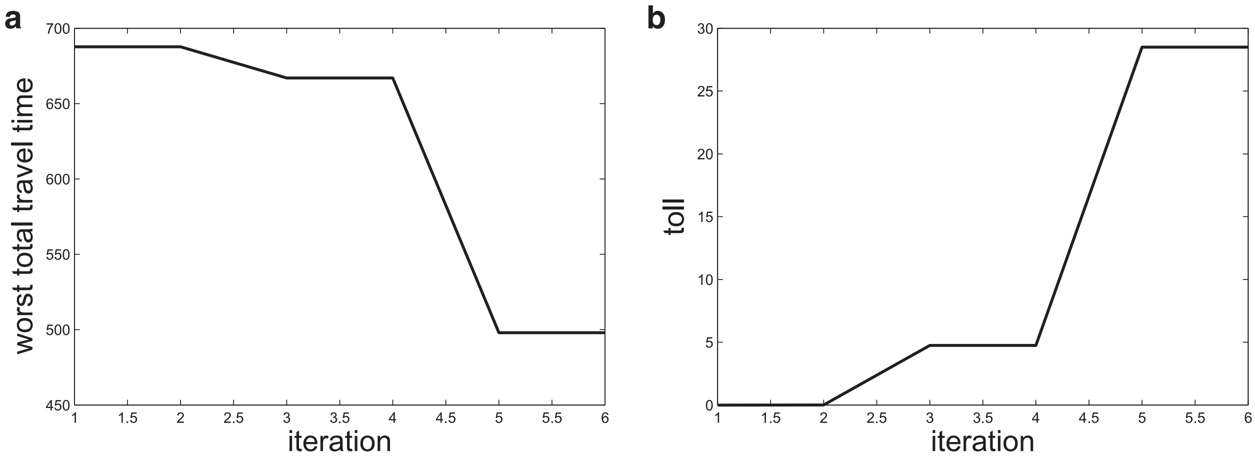


Fig. 8. Numerical results.

quadratic with respect to $\mathbf{f}$ and $\mathcal{F}_k^{\varepsilon}(\mathbf{y})$is characterized by a system of linear inequalities, evaluating $\Phi_k(\mathbf{y}^{(0)})$ is equivalent to solving $K+1$quadratic programs. Then we can obtain $K+1$optimal values. The maximum function value will be $\Phi(\mathbf{y}^{(0)})$.

2. If the value of $\Phi(\mathbf{y}^{(0)})$ can be further reduced, set $n=n+1$and find an improvement direction to obtain $\mathbf{y}^{(1)}$. Because $\Phi(\mathbf{y})$ may be a non-smooth function with respect to $\mathbf{y}$, it is not easy to obtain its derivative with respect to the toll. To update the toll vector, some direct algorithm can be adopted. The feasible toll set defined by Eq. (2.1) is characterized by a system of linear inequalities, there exist numerous algorithms to reduce the feasible regions and find an improved solution, such as fortified-descent simplex algorithm adopted by Ban et al. (2009) or active-set algorithm by Murty (1988).
3. This process will continue iteratively until $|\Phi(\mathbf{y}^{(n)}) - \Phi(\mathbf{y}^{(n-1)})| < \delta$,where δ is a per-defined positive tolerance parameter. Then $\mathbf{y}^* = \mathbf{y}^{(n)}$.

In the numerical example, the toll constraint is: $0 \le y \le 40$. We fix $\varepsilon = 10$. Start with $y^{(0)} = 0$,follow the procedure aforementioned and the active-set algorithm is employed. $\Phi(y)$ is solved by the MPEC solver in GAMS (Rosenthal and Brooke, 2007). Please note that it can also be solved using a quadratic programming solver such as "quadprog" in MATLAB due to affine link cost functions. The optimal toll is $y^* = 28.5$. With this toll, the minimum of the worst-case system travel time $\Phi^* = 498$. When $y \geq y^*$, only path 1 and 4 carry flows, and actually SO is the only BRUE flow pattern. Fig. 8a–b shows the change of the toll and the minimum of the worst-case system travel time. Convergence is reached after five iterations.

The worst total system travel time function versus the toll is illustrated as the thick black envelope in Fig. 7c. We can see that the optimal toll is not unique. For the worst case, the optimal toll is $y^* \geq 26$. The solved optimal toll belongs to this set.

Remark. The proposed algorithm used to solve this example may not be directly applicable for large-scale second-best toll implementations. More efficient algorithms are needed. For example, Lou et al. (2010) proposed a heuristic algorithm using penalization and a cutting-plane scheme to solve the worst-case BRUE tolls and implemented it in a median-size nine-node network. Again, as the algorithm is not the main focus of this paper, we will leave it for future research.

7. Conclusion

This study focuses on developing methodologies of transportation network design problem within the framework of bounded rationality (BR-NDP). Boundedly rational user equilibria (BRUE) is shown to be non-unique and its solution set is not convex. Non-uniqueness of the BRUE flow set poses risk or uncertainty for urban planners while making transportation planning, because they do not know at which equilibrium the network will operate after a plan is implemented. In the established literature, two attitudes towards risk caused by non-uniqueness of the user equilibrium set have been proposed: risk-prone and risk-averse. Each attitude represents one belief that a particular flow pattern within the equilibrium set is most likely to happen. In this study, we also assume urban planners hold the above two attitudes towards risks caused by non-uniqueness of the BRUE set.

Due to the complexity of BR-NDP, congestion pricing problem (the continuous NDP) is studied. Accordingly, we can define three types of toll congestion under the assumption of BR (BR-TP): risk-prone BR-TP (BR-TP-RP), risk-averse BR-TP (BR-TP-RA), and risk-neutral BR-TP (BR-TP-RN). Only two methodologies of BR-TP-RP and BR-TP-RA are discussed. They can be formulated as either a min -min or a min -max program.

Given affine link cost functions, the BRUE set can be decomposed into multiple disjoint convex subsets. Then the lower-level BR-TP-RP can be optimized over each convex subset and its solution existence conditions are established. However, the solution of BR-TP-RA cannot be ensured due to upper semi-continuity of the lower-level problem. Since the lower-level objective function is non-smooth with respect to the toll, it is difficult to evaluate its derivatives. Some direct algorithm is adopted to search for optimal tolls.

Two examples are used to illustrate the proposed methodologies. A small network with three paths is studied. The optimal tolls for BR-TP-RP and BR-TP-RA can be solved analytically. The properties of BR-TP are revealed from analytical solutions. Comparing the three optimal tolls under BR-TP-RP, BR-TP-RA, and BR-TP-UE (only UE is considered in the lower-level), we find out higher tolls are needed when the attitudes change from optimistic to pessimistic. The other network with four paths is solved numerically by adopting the proposed procedure.

Though the solution existence condition of the BR-TP-RP is mathematically validated in this paper, the attainment of the optimal toll for BR-TP-RA cannot be guaranteed and needs to be further explored. In addition, solving the second best congestion pricing under BR with risk-neutral attitude (BR-TP-RN) is another direction and the existing work of risk-neutral congestion under UE (Ban et al., 2013) can be extended to the BR context. The last but not the least, a more practical methodology of the BR-TP in large-scale networks given nonlinear link cost functions should be developed and convergence of the algorithm needs to be further discussed.

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