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A general extended Kalman filter for simultaneous estimation of system and unknown inputs

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ABSTRACT

The traditional Extended Kalman filter (EKF) is a useful tool for structural parameter identification with limited observations. It is, however, not applicable when the excitations on the structure are unknown or the excitation locations are not monitored. A novel Extended Kalman filter approach referred to as the General Extended Kalman filter with unknown inputs (GEKF-UI) is proposed to estimate the structural parameters and the unknown excitations (inputs) simultaneously. The proposed GEKF-UI gives an analytical EKF solution dealing with the more general measurement scenarios with the existing EKF methods as its special cases. Existing constraints on sensor configuration have been removed enabling more general application to complex structures. Simulation results from a 3-storey linear damped shear building, an ASCE benchmark structure and a two-storey planar frame structure are used to validate the proposed method for both time-invariant and time-varying system identification.

1. Introduction

1.1. The simultaneous state and unknown input estimation using Kalman filtering

The Extended Kalman filter (EKF) [1] is a simple but powerful tool for the identification of structural parameters such as stiffness, damping and nonlinear parameters with limited measurements for structural health monitoring (SHM) of structures [2–5]. Existing EKF approaches treat the unknown structural parameters as part of the states to be estimated. To track the evolution of the estimates, all the external excitations (inputs) should be known or measured. However, some inputs may not be measured or known in practice, such as the seismic excitations, the ambient wind loads, the moving traffic loads, etc. Therefore, an EKF approach which can deal with unknown inputs is in need. Though theoretical study on linear filtering with unknown inputs has been extensively studied during the past few decades [6–12], yet their nonlinear counterparts, the EKF method with unknown inputs have not been broadly investigated. Wang and Haldar [13] proposed an iterative least-squares procedure in conjunction with the traditional EKF to estimate the states and unknown inputs off-line with limited observations. Lately, Al-Hussein and Haldar [14] improved the method by replacing the EKF formulation with the more recent unscented Kalman filter (UKF) to obtain improved performance of structural health assessment (SHA). However, the drawbacks in the numerical procedure of the off-line original method [13] are still retained. To track the evolution of the structural parameters (damages of the structures) on-line, an extended Kalman filter with unknown inputs (EKF-UI) was derived analytically by minimizing a least-squares objective function [15]. More recently, a method which sequentially estimates the unknown inputs with the least-squares estimation and the states with the traditional EKF was proposed [16,17] for both linear and nonlinear structures. Other than the identification of external excitations, it has been shown that the EKF with unknown inputs can be extended to a large structural system [16] based on a decomposition method. The intra-connection effect between structural elements is regarded as unknown input to the substructures. The measurements at the sub-structural interface degrees-of-freedom (DOFs) are not necessary as a result, and the number of sensors in the SHM system for a large scale structure is thus reduced. These serve as good justifications for the development of a new EKF approach with unknown inputs.

For the above-mentioned EKF approaches [13–16], all inputs are required to be present in the observation equations, which imposes constraints on the number and location of the sensors required. For

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instance, the sensors should be installed at all the DOFs corresponding to the external excitations in Lei et al. [16]. Moreover, if the measurements are not in the form of accelerations, extra signal processing is needed with additional computation effort. For example, before the application of the EKF-UI approach [15], double numerical differentiations have to be made with the aid of low-pass digital filter to obtain the accelerations from the measured displacements of a 3-storey nonlinear shear building subjected to unknown earthquake. To reduce these restrictions Pan et al. [18] have proposed an EKF with unknown inputs without direct feedthrough (EKF-UI-WDF) approach where all the unknown inputs are in the model equations whereas none of them are explicitly shown in the observation equations.

1.2. Weakness with existing Kalman filtering approaches with unknown inputs

To the best of the authors' knowledge, literature on Kalman Filter with unknown inputs is mainly divided into two categories: (I) those with the unknown inputs included in both the model equations and the observation equations [9,11–17]; and (II) those with the unknown inputs found only in the model equations [6,8,10,18]. The coefficient matrix of unknown inputs in the observation equations of the first group should be of full column rank [15]. The same coefficient matrix for the latter group is a zero matrix, which results in one step delay in the estimation of the unknown inputs.

To demonstrate the differences more clearly, an ASCE benchmark 4-storey structure [19] subjected to two white noise excitations at the 2nd and 4th floors is adopted in the following discussion. Numerical analysis on this example is given in Section 4 of this paper.

By introducing the extended state vector $\mathbf{Z} = \{\mathbf{x}^{\mathrm{T}}, \dot{\mathbf{x}}^{\mathrm{T}}, \theta^{\mathrm{T}}\}^{\mathrm{T}}$ into the equations of motion of the structure, the model equation becomes $\dot{\mathbf{Z}}(t) = \mathbf{g}(\mathbf{Z}, \mathbf{f}^{*}, t)$ where $\mathbf{x}, \dot{\mathbf{x}}, \theta$ are vectors of the storey displacements, velocities and unknown structural parameters, respectively. $\mathbf{f}^{*}(t) = [f_2(t) f_4(t)]^{\mathrm{T}}$ is the unknown input vector. Cases with different measurement scenarios are discussed as follows:

- Case (i) When the measurements are accelerations from all stories at $t = (k+1)\Delta t$ $(k = 0, 1, 2, \ldots)$, i.e., $\mathbf{y}_{k+1} = [\ddot{x}_{1,k+1}\ddot{x}_{2,k+1}\ddot{x}_{3,k+1}\ddot{x}_{4,k+1}]^{\mathrm{T}}$. The observation equations become $\mathbf{y}_{k+1} = \mathbf{h}(\mathbf{Z}_{k+1}, \mathbf{f}^{*}_{k+1}, k+1)$ where $\mathbf{f}^{*}_{k+1} = [f_{2,k+1} f_{4,k+1}]^{\mathrm{T}}$. Since the rank of the coefficient matrix with unknown inputs $\mathbf{D}^{*}_{k+1|k} = [\partial\mathbf{h}_{k+1}/\partial\mathbf{f}^{*}_{k+1}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$ is two and it is of full column rank, the condition of estimation [14] holds. Therefore, Category I EKF approaches are applicable for the solution [15–17].
- Case (ii) When the measurements are displacements from all stories at $t = (k+1)\Delta t$ $(k = 0, 1, 2, \ldots)$, i.e., $\mathbf{y}_{k+1} = [x_{1,k+1}x_{2,k+1}x_{3,k+1}x_{4,k+1}]^{\mathrm{T}}$. The observation equations become $\mathbf{y}_{k+1} = \mathbf{h}(\mathbf{Z}_{k+1}, k+1)$ where no unknown input exists. The rank of $\mathbf{D}^{*}_{k+1|k} = [\partial\mathbf{h}_{k+1}/\partial\mathbf{f}^{*}_{k+1}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ is zero and is not full rank, and the condition of estimation [15] does not hold. The Category I EKF approaches are not applicable as a result while Category II approaches [18] can deal with this case.
- Case (iii) When no sensor is installed on the second floor, i.e., $\mathbf{y}_{k+1} = [x_{1,k+1}x_{3,k+1}x_{4,k+1}]^{\mathrm{T}}$, the rank of $\mathbf{D}^{*}_{k+1|k} = [\partial\mathbf{h}_{k+1}/\partial\mathbf{f}^{*}_{k+1}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$ is unity which is less than full rank. Therefore, both Categories of EKF approaches discussed above are not applicable. In other words, the simultaneous estimation of the states and inputs with this scenario of measurement cannot be solved with any existing EKF approaches.

As a summary,

(a) For the simultaneous estimation of the state and unknown inputs, Cases (i) and (ii) above show that the EKF approaches in different forms are needed for different measurement scenarios, which brings unnecessary inconvenience for the computation.
(b) Case (iii) shows that for a general measurement scenario where only partial inputs are present in the observation equations, with $0 < rank(\mathbf{D}^{*}_{k+1|k}) < r$, where r is the full column rank, no existing EKF approaches is applicable.

In this paper, a novel EKF approach referred to as the General EKF with unknown inputs (GEKF-UI) is proposed to identify structural parameters and unknown inputs simultaneously. It is an EKF approach with partial measurement and unknown inputs covering the case of $0 \leqslant rank(\mathbf{D}^{*}_{k+1|k}) \leqslant r$, which is more general than existing approaches that require $rank(\mathbf{D}^{*}_{k+1|k}) = r$ or $rank(\mathbf{D}^{*}_{k+1|k}) = 0$. There is no need to measure the responses at all of the DOFs on which the unknown inputs are acting as a result.

The recursive solutions of the GEKF-UI are formulated with the least-squares estimation of an extended state vector with the aid of matrix decompositions. The unknown inputs at all time instants and the current states are combined as the extended state vector whose dimension increases with time. The relationship between the extended state vectors at two consecutive time instants is derived to form a recursive solution on the extended state vector. Matrix decomposition is then applied to obtain the extended state vector.

Simulation results from the following examples are used to validate the proposed method: (a) a three-degrees-of-freedom (DOFs) linear damped shear building subjected to an unknown earthquake excitation; (b) the Phase I ASCE benchmark building for structural health monitoring subjected to two unknown white noise excitations; and (c) a 2-storey plane frame with 12 DOFs subjected to unknown inputs with abrupt reduction of stiffness.

The layout of this paper is given as follows. The problem formulation is given in Section 2. The recursive solutions for the proposed GEKF-UI approach are derived and presented in Section 3 with discussions. To verify the proposed method, three numerical examples are presented in Section 4. Section 5 gives the conclusions. Detailed derivations on the solutions in Sections 3 are given in Appendices A and B.

2. Problem formulation

When the external excitations are not measured (unknown), the equations of motion of a m-DOFs linear damped structure can be expressed as:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{F}_c[\dot{\mathbf{x}}(t)] + \mathbf{F}_s[\mathbf{x}(t)] = \boldsymbol{\eta}^{*}\mathbf{f}^{*}(t) + \boldsymbol{\eta}\mathbf{f}(t) \quad (1)$$

where $\mathbf{f}^{*}(t) = [f^{*}_1(t), \ldots, f^{*}_r(t)]^{\mathrm{T}}$ is the set of r-unknown (or unmeasured) excitations; Matrix $\boldsymbol{\eta}^{*}$ is a mapping matrix associated with $\mathbf{f}^{*}(t)$; $\mathbf{f}(t) = [f_1(t), f_2(t), \ldots, f_s(t)]^{\mathrm{T}}$ is the set of s-known (measured) excitations; and $\boldsymbol{\eta}$ is a $(m \times s)$ mapping matrix associated with $\mathbf{f}(t)$.

Consider an extended unknown state vector with a dimension of $(2m+n)$,

$$\mathbf{Z}(t) = \{\mathbf{x}^{\mathrm{T}}, \dot{\mathbf{x}}^{\mathrm{T}}, \theta^{\mathrm{T}}\}^{\mathrm{T}} \quad (2)$$

where $\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_n]^{\mathrm{T}}$ is a n-unknown parameter vector with θ_i ($i = 1, 2, \ldots, n$) being the ith unknown parameter of the structure, such as damping, stiffness, nonlinear and hysteretic parameters. When $\dot{\theta}_i = 0$ ($i = 1, 2, \ldots, n$) is assumed, i.e., the unknown parameters are constants, and one can transform Eq. (1) into a nonlinear state equation as

$$\dot{\mathbf{Z}}(t) = \mathbf{g}(\mathbf{Z}, \mathbf{f}, \mathbf{f}^*, t) + \mathbf{w}(t) \tag{3}$$

where $\mathbf{Z}(t)$, $\mathbf{f}(t)$, and $\mathbf{f}^*(t)$ are $(2m+n)$-state vector, s-known input vector and$r$-unknown input vector, respectively, $\mathbf{w}(t)$ is the model noise vector (uncertainty) with zero mean and a covariance matrix $\mathbf{Q}(t)$. The discretized observation vector (measured responses) with a dimension of l can also be expressed as

$$\mathbf{y}_{k+1} = \mathbf{h}(\mathbf{Z}_{k+1}, \mathbf{f}_{k+1}, \mathbf{f}^*_{k+1}, k+1) + \mathbf{v}_{k+1} \tag{4}$$

where $\mathbf{y}_{k+1}, \mathbf{Z}_{k+1}, \mathbf{f}_{k+1}$ and $\mathbf{f}^*_{k+1}$ are the l-observation (measured) output vector, $(2m+n)$-state vector, s-known input vector and$r$-unknown input vector respectively at $t = (k+1)\Delta t$ respectively and Δt is the sampling time interval. $\mathbf{v}_{k+1}$ is a Gaussian measurement noise vector with zero mean and a covariance matrix $\mathrm{E}[\mathbf{v}_k \mathbf{v}_j^T] = \mathbf{R}_k \delta_{kj}$ where δ_{kj} is the Kroneker delta.

Both the equation of motion in Eq. (3) and the observation equation in Eq. (4) are nonlinear in the unknown state vector $\mathbf{Z}$ and the unknown input vector $\mathbf{f}^*$. These equations will be linearized with respect to the estimated and predicted state vectors at $t = k\Delta t$ and $t = (k+1)\Delta t, \hat{\mathbf{Z}}_{k|k}$ and $\hat{\mathbf{Z}}_{k+1|k}$ respectively, and the estimated unknown input vector at $t = k\Delta t,\ \hat{\mathbf{f}}^*_{k|k}$, as follows,

$$\mathbf{g}_k \approx \hat{\mathbf{g}}_{k|k} + \mathbf{G}_{k|k}(\mathbf{Z}_k - \hat{\mathbf{Z}}_{k|k}) + \mathbf{B}^*_{k|k}(\mathbf{f}^*_k - \hat{\mathbf{f}}^*_{k|k}) \tag{5}$$

$$\mathbf{h}_{k+1} \approx \hat{\mathbf{h}}_{k+1|k} + \mathbf{H}_{k+1|k}(\mathbf{Z}_{k+1} - \hat{\mathbf{Z}}_{k+1|k}) + \mathbf{D}^*_{k+1|k}(\mathbf{f}^*_{k+1} - \hat{\mathbf{f}}^*_{k|k}) \tag{6}$$

in which the Jacobian matrices $\mathbf{G}_{k|k}, \mathbf{B}^*_{k|k}, \mathbf{H}_{k+1|k}\mathbf{D}^*_{k+1|k}$ are given by

$$\mathbf{G}_{k|k} = \left[\partial \mathbf{g}_k / \partial \mathbf{Z}_k\right]_{\mathbf{Z}_k = \hat{\mathbf{Z}}_{k|k}, \mathbf{f}^*_k = \hat{\mathbf{f}}^*_{k|k}} = \begin{bmatrix} \frac{\partial \hat{g}_{1,k}}{\partial z_{1,k}} & \frac{\partial \hat{g}_{1,k}}{\partial z_{2,k}} & \cdots & \frac{\partial \hat{g}_{1,k}}{\partial z_{2m+n,k}} \\ \frac{\partial \hat{g}_{2,k}}{\partial z_{1,k}} & \frac{\partial \hat{g}_{2,k}}{\partial z_{2,k}} & \cdots & \frac{\partial \hat{g}_{2,k}}{\partial z_{2m+n,k}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \hat{g}_{2m,k}}{\partial z_{1,k}} & \frac{\partial \hat{g}_{2m,k}}{\partial z_{2,k}} & \cdots & \frac{\partial \hat{g}_{2m,k}}{\partial z_{2m+n,k}} \\ \hdashline & \mathbf{0}_{n\times(2m+n)} & & \end{bmatrix} \tag{7}$$

$$\mathbf{B}^*_{k|k} = \left[\partial \mathbf{g}_k / \partial \mathbf{f}^*_k\right]_{\mathbf{Z}_k = \hat{\mathbf{Z}}_{k|k}, \mathbf{f}^*_k = \hat{\mathbf{f}}^*_{k|k}} = \begin{bmatrix} \frac{\partial \hat{g}_{1,k}}{\partial f^*_{1,k}} & \frac{\partial \hat{g}_{1,k}}{\partial f^*_{2,k}} & \cdots & \frac{\partial \hat{g}_{1,k}}{\partial f^*_{r,k}} \\ \frac{\partial \hat{g}_{2,k}}{\partial f^*_{1,k}} & \frac{\partial \hat{g}_{2,k}}{\partial f^*_{2,k}} & \cdots & \frac{\partial \hat{g}_{2,k}}{\partial f^*_{r,k}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \hat{g}_{2m,k}}{\partial f^*_{1,k}} & \frac{\partial \hat{g}_{2m,k}}{\partial f^*_{2,k}} & \cdots & \frac{\partial \hat{g}_{2m,k}}{\partial f^*_{r,k}} \\ \hdashline & \mathbf{0}_{n\times r} & & \end{bmatrix} \tag{8}$$

$$\mathbf{H}_{k+1|k} = [\partial \mathbf{h}_{k+1} / \partial \mathbf{Z}_{k+1}]_{\mathbf{Z}_{k+1} = \hat{\mathbf{Z}}_{k+1|k}, \mathbf{f}^*_{k+1} = \hat{\mathbf{f}}^*_{k|k}} = \begin{bmatrix} \frac{\partial \hat{h}_{1,k+1}}{\partial z_{1,k+1}} & \frac{\partial \hat{h}_{1,k+1}}{\partial z_{2,k+1}} & \cdots & \frac{\partial \hat{h}_{1,k+1}}{\partial z_{2m+n,k+1}} \\ \frac{\partial \hat{h}_{2,k+1}}{\partial z_{1,k+1}} & \frac{\partial \hat{h}_{2,k+1}}{\partial z_{2,k+1}} & \cdots & \frac{\partial \hat{h}_{2,k+1}}{\partial z_{2m+n,k+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \hat{h}_{l,k+1}}{\partial z_{1,k+1}} & \frac{\partial \hat{h}_{l,k+1}}{\partial z_{2,k+1}} & \cdots & \frac{\partial \hat{h}_{l,k+1}}{\partial z_{2m+n,k+1}} \end{bmatrix} \tag{9}$$

$$\mathbf{D}^*_{k+1|k} = [\partial \mathbf{h}_{k+1} / \partial \mathbf{f}^*_{k+1}]_{\mathbf{Z}_{k+1} = \hat{\mathbf{Z}}_{k+1|k}, \mathbf{f}^*_{k+1} = \hat{\mathbf{f}}^*_{k|k}} = \begin{bmatrix} \frac{\partial \hat{h}_{1,k+1}}{\partial f^*_{1,k+1}} & \frac{\partial \hat{h}_{1,k+1}}{\partial f^*_{2,k+1}} & \cdots & \frac{\partial \hat{h}_{1,k+1}}{\partial f^*_{r,k+1}} \\ \frac{\partial \hat{h}_{2,k+1}}{\partial f^*_{1,k+1}} & \frac{\partial \hat{h}_{2,k+1}}{\partial f^*_{2,k+1}} & \cdots & \frac{\partial \hat{h}_{2,k+1}}{\partial f^*_{r,k+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \hat{h}_{l,k+1}}{\partial f^*_{1,k+1}} & \frac{\partial \hat{h}_{l,k+1}}{\partial f^*_{2,k+1}} & \cdots & \frac{\partial \hat{h}_{l,k+1}}{\partial f^*_{r,k+1}} \end{bmatrix} \tag{10}$$

with

$$\mathbf{g}_k = \mathbf{g}(\mathbf{Z}_k, \mathbf{f}_k, \mathbf{f}^*_k, k\Delta t), \quad \mathbf{h}_{k+1} = \mathbf{h}(\mathbf{Z}_{k+1}, \mathbf{f}_{k+1}, \mathbf{f}^*_{k+1}, k+1)$$

$$\hat{\mathbf{g}}_{k|k} = \mathbf{g}(\hat{\mathbf{Z}}_{k|k}, \mathbf{f}_k, \hat{\mathbf{f}}^*_{k|k}, \mathrm{k}\Delta t), \quad \hat{\mathbf{h}}_{k+1|k} = \mathbf{h}(\hat{\mathbf{Z}}_{k+1|k}, \mathbf{f}_{k+1}, \hat{\mathbf{f}}^*_{k|k}, k+1)$$

and

$$\frac{\partial \hat{g}_{i,k}}{\partial z_{j,k}} = \left.\frac{\partial g_{i,k}}{\partial z_{j,k}}\right|_{\mathbf{Z}_k = \hat{\mathbf{Z}}_{k|k}, \mathbf{f}^*_k = \hat{\mathbf{f}}^*_{k|k}}, \quad \frac{\partial \hat{g}_{i,k}}{\partial f^*_{q,k}} = \left.\frac{\partial g_{i,k}}{\partial f^*_{q,k}}\right|_{\mathbf{Z}_k = \hat{\mathbf{Z}}_{k|k}, \mathbf{f}^*_k = \hat{\mathbf{f}}^*_{k|k}}$$

$$\frac{\partial \hat{h}_{p,k+1}}{\partial z_{j,k+1}} = \left.\frac{\partial h_{p,k+1}}{\partial z_{j,k+1}}\right|_{\mathbf{Z}_{k+1} = \hat{\mathbf{Z}}_{k+1|k}, \mathbf{f}^*_{k+1} = \hat{\mathbf{f}}^*_{k|k}}, \quad \frac{\partial \hat{h}_{p,k+1}}{\partial f^*_{q,k+1}} = \left.\frac{\partial h_{p,k+1}}{\partial f^*_{q,k+1}}\right|_{\mathbf{Z}_{k+1} = \hat{\mathbf{Z}}_{k+1|k}, \mathbf{f}^*_{k+1} = \hat{\mathbf{f}}^*_{k|k}}$$

where $g_{i,k}$ is the ith component of the vector $\mathbf{g}_k$ for $i = 1, 2, \ldots, 2m, \hat{h}_{p,k+1}$ is the pth component of the vector $\mathbf{h}_{k+1}$ for $p = 1, 2, \ldots, l, z_{j,k}$ and $z_{j,k+1}$ are the jth components of the state vectors $\mathbf{Z}_k$ and $\mathbf{Z}_{k+1}$ respectively for $j = 1, 2, \ldots, 2m+n$, and $f^*_{q,k}$ and $f^*_{q,k+1}$ are the qth components of the unknown input vectors $\mathbf{f}^*_k$ and $\mathbf{f}^*_{k+1}$ respectively for $q = 1, 2, \ldots, r$.

It is observed from Eqs. (7)–(10) that the Jacobian matrices $\mathbf{G}_{k|k}, \mathbf{B}^*_{k|k}, \mathbf{H}_{k+1|k}\mathbf{D}^*_{k+1|k}$ are obtained by calculating the first order partial derivative of the vectors $\mathbf{g}_k$ and $\mathbf{h}_{k+1}$ with respect to the predicted states and unknown inputs at $t = (k+1)\Delta t$. This requires the analytical first derivatives of the component of $\mathbf{g}_k$ and $\mathbf{h}_{k+1}$ with high dimensions, say, more than 10, and this is very time consuming. Moreover, the inaccuracy of the calculation of the above Jacobian matrices may result in the divergence and instability of the EKF methods. To overcome this problem, a reasonably small sampling time interval Δt is required, e.g., $\Delta t = 0.001$ s [14]. Also, taking the numerical integral of the nonlinear function $\mathbf{g}(\mathbf{Z}, \mathbf{f}, \mathbf{f}^*, t)$ in Eq. (3) over the time interval $[k\Delta t, (k+1)\Delta t]$ to obtain a more accurate predicted state vector $\hat{\mathbf{Z}}_{k+1|k}$ (see [2,15,18]) may help to improve the computation accuracy of $\mathbf{G}_{k|k}, \mathbf{B}^*_{k|k}, \mathbf{H}_{k+1|k}\mathbf{D}^*_{k+1|k}$. Both of the above computation techniques have been used in the implementation of the proposed GEKF-UI in this paper. This will of course incur heavy computation inevitably due to the higher sampling frequency and the numerical integration at each time step.

Substituting $\dot{\mathbf{Z}}(k\Delta t) = (\mathbf{Z}_{k+1} - \mathbf{Z}_k)(\Delta t)^{-1}$ into the left side of Eq. (3), and together with Eqs. (5)–(10), the discrete state observation equations can be written as

$$\mathbf{Z}_{k+1} = \mathbf{\Phi}_{k+1,k}\mathbf{Z}_k + \mathbf{B}^*_k \mathbf{f}^*_k + \mathbf{u}_{k|k} + \mathbf{w}_{k+1} \tag{11}$$

$$\mathbf{y}_{k+1} = \mathbf{H}_{k+1|k}\mathbf{Z}_{k+1} + \mathbf{D}^*_{k+1|k}\mathbf{f}^*_{k+1} + \tilde{\mathbf{u}}_{k+1|k} + \mathbf{v}_{k+1} \tag{12}$$

where $\mathbf{\Phi}_{k+1,k} = \mathbf{I} + \Delta t\ \mathbf{G}_{k|k}$ is the state transition matrix, $\mathbf{B}^*_k = \Delta t \mathbf{B}^*_{k|k}$ and

$$\mathbf{u}_{k|k} = [\mathbf{g}(\hat{\mathbf{Z}}_{k|k}, \mathbf{f}_k, \hat{\mathbf{f}}^*_{k|k}, \mathrm{k}\Delta t) - \mathbf{G}_{k|k}\hat{\mathbf{Z}}_{k|k} - \mathbf{B}^*_{k|k}\hat{\mathbf{f}}^*_{k|k}]\Delta t \tag{13}$$

$$\tilde{\mathbf{u}}_{k+1|k} = \hat{\mathbf{h}}_{k+1|k} - \mathbf{H}_{k+1|k}\hat{\mathbf{Z}}_{k+1|k} - \mathbf{D}^*_{k+1|k}\hat{\mathbf{f}}^*_{k|k} \tag{14}$$

When no entries of the unknown inputs are present in the observation equation (i.e., $\mathbf{D}^*_{k+1|k} = \mathbf{0}$ or $rank(\mathbf{D}^*_{k+1|k}) = 0$), Eq. (12) becomes

$$\mathbf{y}_{k+1} = \mathbf{H}_{k+1|k}\mathbf{Z}_{k+1} + \tilde{\mathbf{u}}_{k+1|k} \tag{15}$$

It is noted that the unknown inputs with one step delay, $\mathbf{f}_k^*$, instead of $\mathbf{f}_{k+1}^*$, are involved in the measurements $\mathbf{y}_{k+1}$ of Eq. (15). Based on Eqs. (11) and (15), an EKF-UI-WDF approach has been proposed by Pan et al. [18] to obtain the estimates for the states $\mathbf{Z}_{k+1}$ and the unknown inputs $\mathbf{f}_k^*$ at $t = (k+1)\Delta t$, i.e., $\hat{\mathbf{Z}}_{k+1|k+1}$ and $\hat{\mathbf{f}}_{k|k+1}^*$, respectively. There is no doubt that the cases with $rank(\mathbf{D}_{k+1|k}^*) = r$ [14] and $rank(\mathbf{D}_{k+1|k}^*) = 0$ [18] are special cases of the general problem $0 \leqslant rank(\mathbf{D}_{k+1|k}^*) \leqslant r$ where some entries of the unknown inputs are missing in the observation equations and the corresponding columns of $\mathbf{D}_{k+1|k}^*$ in Eq. (12) become zeros. However, to the best knowledge of the authors, no report is found in the literature on the solutions of this more general problem.

The general problem $0 \leqslant rank(\mathbf{D}_{k+1|k}^*) \leqslant r$ will be investigated below with formulation of a novel EKF approach with the unknown inputs. Inspired by the formulation of the EKF-UI-WDF [18] where an unknown input is not present in the observation equation, the estimation problem has one-step delay from the current measurement time i.e., $\hat{\mathbf{f}}_{k|k+1}^*$ instead of $\mathbf{f}_k^*$ is estimated at $t = (k+1)\Delta t$. The unknown input vector to be estimated at $t = (k+1)\Delta t$ for the general problem is chosen as

$$\mathbf{f}_{e,k+1}^* = \left[\tilde{\mathbf{f}}_k^{*\mathrm{T}} \;\vdots\; \bar{\mathbf{f}}_{k+1}^{*\mathrm{T}}\right]^{\mathrm{T}} \tag{16}$$

where $\tilde{\mathbf{f}}_k^* = \tilde{\mathbf{f}}^*(t = k\Delta t)$ is the $\tilde{r}$-unknown input vector at $t = k\Delta t$ which is not in the observation equations but in the model equations. $\bar{\mathbf{f}}_{k+1}^* = \bar{\mathbf{f}}^*(t = (k+1)\Delta t)$ is the $\bar{r}$-unknown input vector which is found in both the observation and model equations at $t = (k+1)\Delta t$. (It should be noted that $\tilde{\mathbf{f}}^*(t)$ and $\bar{\mathbf{f}}^*(t)$ are sub-vectors of the unknown input vector $\mathbf{f}^*(t)$ in Eq. (3), i.e., $\mathbf{f}^*(t) = \left[\tilde{\mathbf{f}}^{*\mathrm{T}}(t) \quad \bar{\mathbf{f}}^{*\mathrm{T}}(t)\right]^{\mathrm{T}}$, with the dimension relationship $r = \tilde{r} + \bar{r}$). Correspondingly, the coefficient matrix of the unknown inputs $\mathbf{D}_{k+1|k}^*$ in Eq. (6) is partitioned as

$$\mathbf{D}_{k+1|k}^* = \left[\mathbf{0}_{l\times\tilde{r}} \;\vdots\; \bar{\mathbf{D}}_{k+1|k}^*\right] \tag{17}$$

where $\bar{\mathbf{D}}_{k+1|k}^*$ is a $(l \times \bar{r})$ matrix with full column rank $\bar{r}$.

Let $\hat{\mathbf{Z}}_{k+1|k+1}$ and $\hat{\mathbf{f}}_{e,k+1|k+1}^* = \left[\hat{\tilde{\mathbf{f}}}_{k|k+1}^{*\mathrm{T}} \;\vdots\; \hat{\bar{\mathbf{f}}}_{k+1|k+1}^{*\mathrm{T}}\right]^{\mathrm{T}}$ be the estimates of $\mathbf{Z}_{k+1}$ and $\mathbf{f}_{e,k+1}^*$ at $t = (k+1)\Delta t$, respectively. The problem to be solved is then summarized as: The recursive solutions of $\hat{\mathbf{Z}}_{k+1|k+1}$ and $\hat{\mathbf{f}}_{e,k+1|k+1}^*$ are estimated given the observations $(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{k+1})$.

3. The recursive solutions for the proposed GEKF-UI approach

A least-squares objective function J_{k+1} is given as follows:

$$\mathrm{J}_{k+1} = \sum_{i=1}^{k+1} \Delta_i^{\mathrm{T}} \mathbf{R}_i^{-1} \Delta_i = \bar{\Delta}_{k+1}^{\mathrm{T}} \mathbf{W}_{k+1} \bar{\Delta}_{k+1} \tag{18}$$

where

$$\Delta_i = \mathbf{y}_i - \mathbf{h}(\mathbf{Z}_i, \mathbf{f}_i, \mathbf{f}_i^*, i); \quad \bar{\Delta}_{k+1} = [\Delta_1^{\mathrm{T}}, \Delta_2^{\mathrm{T}}, \ldots, \Delta_{k+1}^{\mathrm{T}}]^{\mathrm{T}} \tag{19}$$

and Δ_i is the l-error vector between the measurement $\mathbf{y}_i$ and the linearized output value $\mathbf{h}(\mathbf{Z}_i, \mathbf{f}_i, \mathbf{f}_i^*, i)$ at $t = i\Delta t$. Vector $\mathbf{h}(\mathbf{Z}_i, \mathbf{f}_i, \mathbf{f}_i^*, i)$ can be obtained from Eq. (6) by replacing $k+1$ with i ($i = 1, 2, \ldots, k+1$). $\mathbf{W}_{k+1}$ is a $[l(k+1)\times l(k+1)]$ diagonal matrix with $\mathbf{R}_i^{-1}$ as the main diagonal components i.e., $\mathbf{W}_{k+1} = diag(\mathbf{R}_1^{-1}, \mathbf{R}_2^{-1}, \ldots, \mathbf{R}_{k+1}^{-1}), \mathbf{R}_i = E[\mathbf{v}_i\mathbf{v}_i^{\mathrm{T}}]$ and $\mathbf{v}_i$ is the l-measurement noise vector at $t = i\Delta t$.

The objective function J_{k+1} is a quadratic function of the unknown extended vector $\mathbf{Z}_{e,k+1}$. Assuming that the dimension of the observation vector, l, is greater than the number of unknown excitations, r, i.e., $l > r$, the estimated $\hat{\mathbf{Z}}_{e,k+1|k+1}$ of $\mathbf{Z}_{e,k+1}$ at $t = (k+1)\Delta t$ can be obtained by minimizing J_{k+1} as

$$\hat{\mathbf{Z}}_{e,k+1|k+1} = \mathbf{P}_{e,k+1}[\mathbf{A}_{e,k+1}^{\mathrm{T}}\mathbf{W}_{k+1}\mathbf{Y}_{k+1}]; \quad \mathbf{P}_{e,k+1} = [\mathbf{A}_{e,k+1}^{\mathrm{T}}\mathbf{W}_{k+1}\mathbf{A}_{e,k+1}]^{-1} \tag{20a}$$

Detailed derivations of matrices $\mathbf{Y}_{k+1}$ and $\mathbf{A}_{e,k+1}^{\mathrm{T}}$ can be found in Appendix A. With the observations $(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{k+1})$,

$$\begin{aligned}\hat{\mathbf{Z}}_{e,k+1|k+1} &= \left[\hat{\mathbf{Z}}_{k+1|k+1}^{\mathrm{T}} \;\vdots\; \hat{\bar{\mathbf{f}}}_{1|k+1}^{*\mathrm{T}} \;\vdots\; \hat{\tilde{\mathbf{f}}}_{1|k+1}^{*\mathrm{T}} \quad \hat{\bar{\mathbf{f}}}_{2|k+1}^{*\mathrm{T}} \;\vdots\; \cdots \;\vdots\; \hat{\tilde{\mathbf{f}}}_{k|k+1}^{*\mathrm{T}} \quad \hat{\bar{\mathbf{f}}}_{k+1|k+1}^{*\mathrm{T}}\right]^{\mathrm{T}} \\ &= \left[\hat{\mathbf{Z}}_{k+1|k+1}^{\mathrm{T}} \;\vdots\; \hat{\bar{\mathbf{f}}}_{1|k+1}^{*\mathrm{T}} \;\vdots\; \hat{\mathbf{f}}_{e,2|k+1}^{*\mathrm{T}} \;\vdots\; \cdots \;\vdots\; \hat{\mathbf{f}}_{e,k+1|k+1}^{*\mathrm{T}}\right]^{\mathrm{T}}\end{aligned} \tag{20b}$$

is the estimate of $\mathbf{Z}_{e,k+1}$. Appendix A is referred.

Detailed derivations of the recursive solutions of the least-squares estimate of $\mathbf{Z}_{k+1}$ and $\mathbf{f}_{e,k+1}^*$, i.e. $\hat{\mathbf{Z}}_{k+1|k+1}$ and $\hat{\mathbf{f}}_{e,k+1|k+1}^* = \left[\hat{\tilde{\mathbf{f}}}_{k|k+1}^{*\mathrm{T}} \;\vdots\; \hat{\bar{\mathbf{f}}}_{k+1|k+1}^{*\mathrm{T}}\right]^{\mathrm{T}}$ are shown in Appendices A and B and they are given as follows:

$$\hat{\mathbf{Z}}_{k+1|k+1} = \hat{\mathbf{Z}}_{k+1|k} + \mathbf{K}_{\mathbf{Z},k+1}[\mathbf{y}_{k+1} - \mathbf{h}(\hat{\mathbf{Z}}_{k+1|k}, \mathbf{f}_k, \hat{\bar{\mathbf{f}}}_{k+1|k+1}^*, k+1)] \tag{21}$$

$$\begin{aligned}\hat{\tilde{\mathbf{f}}}_{k|k+1}^* &= \mathbf{S}_{k+1,12}\bar{\mathbf{D}}_{k+1|k}^{*\mathrm{T}}\mathbf{R}_{k+1}^{-1}(\mathbf{I}_l - \mathbf{H}_{k+1|k}\mathbf{K}_{\mathbf{Z},k+1})(\mathbf{y}_{k+1} - \hat{\mathbf{h}}_{k+1|k} \\ &\quad + \bar{\mathbf{D}}_{k+1|k}^*\hat{\bar{\mathbf{f}}}_{k|k}^*) + \mathbf{S}_{k+1,11}\tilde{\mathbf{B}}_k^{*\mathrm{T}}\mathbf{H}_{k+1|k}^{\mathrm{T}}(\mathbf{I}_l - \mathbf{H}_{k+1|k}\mathbf{K}_{\mathbf{Z},k+1})(\mathbf{y}_{k+1} \\ &\quad - \hat{\mathbf{h}}_{k+1|k} + \mathbf{H}_{k+1|k}\tilde{\mathbf{B}}_k^*\hat{\tilde{\mathbf{f}}}_{k-1|k}^*)\end{aligned} \tag{22}$$

$$\begin{aligned}\hat{\bar{\mathbf{f}}}_{k+1|k+1}^* &= \mathbf{S}_{k+1,22}\bar{\mathbf{D}}_{k+1|k}^{*\mathrm{T}}\mathbf{R}_{k+1}^{-1}(\mathbf{I}_l - \mathbf{H}_{k+1|k}\mathbf{K}_{\mathbf{Z},k+1})(\mathbf{y}_{k+1} - \hat{\mathbf{h}}_{k+1|k} \\ &\quad + \bar{\mathbf{D}}_{k+1|k}^*\hat{\bar{\mathbf{f}}}_{k|k}^*) + \mathbf{S}_{k+1,12}\tilde{\mathbf{B}}_k^{*\mathrm{T}}\mathbf{H}_{k+1|k}^{\mathrm{T}}(\mathbf{I}_l - \mathbf{H}_{k+1|k}\mathbf{K}_{\mathbf{Z},k+1}) \\ &\quad \times (\mathbf{y}_{k+1} - \hat{\mathbf{h}}_{k+1|k} + \mathbf{H}_{k+1|k}\tilde{\mathbf{B}}_k^*\hat{\tilde{\mathbf{f}}}_{k-1|k}^*)\end{aligned} \tag{23}$$

where

$$\hat{\mathbf{Z}}_{k+1|k} = \hat{\mathbf{Z}}_{k|k} + \int_{\mathrm{k\Delta t}}^{\mathrm{(k+1)\Delta t}} \mathbf{g}(\hat{\mathbf{Z}}_{t|k}, \mathbf{f}, \hat{\mathbf{f}}_{t|k}^*, t)dt; \quad \hat{\mathbf{f}}_{t|k}^* = \left[\hat{\tilde{\mathbf{f}}}_{t|k}^* \quad \hat{\bar{\mathbf{f}}}_{t|k}^*\right] \tag{24}$$

$$\mathbf{K}_{\mathbf{Z},k+1} = \mathbf{P}_{\mathbf{Z},k+1|k}\mathbf{H}_{k+1|k}^{\mathrm{T}}[\mathbf{H}_{k+1|k}\mathbf{P}_{\mathbf{Z},k+1|k}\mathbf{H}_{k+1|k}^{\mathrm{T}} + \mathbf{R}_{k+1}]^{-1} \tag{25}$$

$$\mathbf{S}_{k+1} = \begin{bmatrix}\mathbf{S}_{k+1,11} & \mathbf{S}_{k+1,12} \\ \mathbf{S}_{k+1,12} & \mathbf{S}_{k+1,22}\end{bmatrix} = \begin{bmatrix}\tilde{\mathbf{B}}_k^{*\mathrm{T}}\mathbf{H}_{k+1|k}^{\mathrm{T}}\mathbf{R}_{k+1}^{-1}(\mathbf{I}_l - \mathbf{H}_{k+1|k}\mathbf{K}_{\mathbf{Z},k+1})\mathbf{H}_{k+1|k}\tilde{\mathbf{B}}_k^* & \tilde{\mathbf{B}}_k^{*\mathrm{T}}\mathbf{H}_{k+1|k}^{\mathrm{T}}\mathbf{R}_{k+1}^{-1}(\mathbf{I}_l - \mathbf{H}_{k+1|k}\mathbf{K}_{\mathbf{Z},k+1})\bar{\mathbf{D}}_{k+1|k}^* \\ \bar{\mathbf{D}}_{k+1|k}^{*\mathrm{T}}\mathbf{R}_{k+1}^{-1}(\mathbf{I}_l - \mathbf{H}_{k+1|k}\mathbf{K}_{\mathbf{Z},k+1})\mathbf{H}_{k+1|k}\tilde{\mathbf{B}}_k^* & \bar{\mathbf{D}}_{k+1|k}^{*\mathrm{T}}\mathbf{R}_{k+1}^{-1}(\mathbf{I}_l - \mathbf{H}_{k+1|k}\mathbf{K}_{\mathbf{Z},k+1})\bar{\mathbf{D}}_{k+1|k}^*\end{bmatrix}^{-1} \tag{26}$$

Matrix $\mathbf{S}_{k+1}$ is the error covariance matrix of the estimated unknown excitation vector $\hat{\mathbf{f}}^*_{e,k+1|k+1}$, with $\mathbf{S}_{k+1} = E\{(\mathbf{f}^*_{e,k+1} - \hat{\mathbf{f}}^*_{e,k+1|k+1})(\mathbf{f}^*_{e,k+1} - \hat{\mathbf{f}}^*_{e,k+1|k+1})^{\mathrm{T}}\}$. The full derivation of this matrix is given in Appendix B.

Matrix $\tilde{\mathbf{B}}^*_k$ above is the $[(2m+n)\times\tilde{r}]$ sub-matrix of the input coefficient matrix $\mathbf{B}^*_k$ with full column rank $\tilde{r}$, i.e., $\mathbf{B}^*_k = \begin{bmatrix}\tilde{\mathbf{B}}^*_k & \bar{\mathbf{B}}^*_k\end{bmatrix}$ with $\mathbf{B}^*_k = \Delta t\mathbf{B}^*_{k|k}$. Vector $\hat{\mathbf{Z}}_{t|k}$ in Eq. (24) is the solution of Eq. (3) within time interval $(k\Delta t \leqslant t \leqslant (k+1)\Delta t)$ with the initial condition $\hat{\mathbf{Z}}_{k|k}$ and $\mathbf{w} = \mathbf{0}$. Matrix $\hat{\mathbf{f}}^*_{t|k}$ is the estimate of $\mathbf{f}^*(t)$ within interval $(k\Delta t \leqslant t \leqslant (k+1)\Delta t)$. Vector $\hat{\mathbf{h}}_{k+1|k}$ represents $\mathbf{h}(\hat{\mathbf{Z}}_{k+1|k}, \mathbf{f}_{k+1}, \hat{\hat{\mathbf{f}}}^*_{k|k}, k+1)$ in Eqs. (22) and (23) and $\mathbf{K}_{\mathbf{Z},k+1}$ is the Kalman gain matrix, and $\mathbf{H}_{k+1|k}$ is given by Eq. (9). $\bar{\mathbf{D}}^*_{k+1|k}$ is given by Eq. (17), and matrix $\mathbf{P}_{\mathbf{Z},k+1|k} = \mathrm{E}\{(\mathbf{Z}_{k+1} - \hat{\mathbf{Z}}_{k+1|k})(\mathbf{Z}_{k+1} - \hat{\mathbf{Z}}_{k+1|k})^{\mathrm{T}}\}$ is the state error covariance matrix of $\hat{\mathbf{Z}}_{k+1|k}$ given by

$$\mathbf{P}_{\mathbf{Z},k+1|k} = \mathbf{\Phi}_{k+1,k}\mathbf{P}_{\mathbf{Z},k|k}\mathbf{\Phi}^{\mathrm{T}}_{k+1,k} + \mathbf{Q}_{k+1} \tag{27}$$

where $\mathbf{Q}_{k+1}$ is the variance matrix of the model error at $t = (k+1)\Delta t, \mathbf{\Phi}_{k+1,k} = \mathbf{I} + \Delta t\mathbf{G}_{k|k}$ is the state transition matrix of the linearized system. According to the discussion of Jazwinski [1], the matrix $\mathbf{P}_{\mathbf{Z},k+1|k+1}$ is the error covariance matrix of $\hat{\mathbf{Z}}_{k+1|k+1}$, with $\mathbf{P}_{\mathbf{Z},k+1|k+1} = \mathrm{E}\{(\mathbf{Z}_{k+1} - \hat{\mathbf{Z}}_{k+1|k+1})(\mathbf{Z}_{k+1} - \hat{\mathbf{Z}}_{k+1|k+1})^{\mathrm{T}}\}$. $\mathbf{P}_{\mathbf{Z},k|k}$ in Eq. (27) is given by

$$\begin{aligned}\mathbf{P}_{\mathbf{Z},k|k} = (\mathbf{I} - \mathbf{K}_{\mathbf{Z},k}\mathbf{H}_{k|k-1})\{\mathbf{P}_{\mathbf{Z},k|k-1} + \begin{bmatrix}\tilde{\mathbf{B}}^*_{k-1} & \mathbf{H}^{\mathrm{T}}_{k|k-1}\mathbf{R}^{-1}_k\bar{\mathbf{D}}^*_{k|k-1}\end{bmatrix} \\ \times \mathbf{S}_k\begin{bmatrix}\tilde{\mathbf{B}}^*_{k-1} & \mathbf{H}^{\mathrm{T}}_{k|k-1}\mathbf{R}^{-1}_k\bar{\mathbf{D}}^*_{k|k-1}\end{bmatrix}^{\mathrm{T}}(\mathbf{I} - \mathbf{K}_{\mathbf{Z},k}\mathbf{H}_{k|k-1})^{\mathrm{T}}\}\end{aligned} \tag{28}$$

where $\mathbf{K}_{\mathbf{Z},k}$ and $\mathbf{S}_k$ are obtained respectively from Eqs. (25) and (26) by replacing $k+1$ with k. The analytical solutions derived in Eqs. (21)–(28) is referred to as the General Extended Kalman Filter with Unknown Inputs (GEKF-UI) which is new and original.

Further remarks:

(1) Configuration of the proposed GEKF-UI approach
Similar to the traditional Kalman filter [20], the proposed GEKF-UI approach can be divided into two parts—the Measurement Update (Correction) and the Time Update (Prediction). In the first part, Eqs. (21)–(23) and (25) are on the estimation of states with the unknown inputs updated by the current measurements as well as the gain matrix for the states. Eqs. (26) and (28) are on the error covariance matrices for the unknown inputs with the states updated by current measurements. In the second part, Eqs. (24) and (27) represent the estimation and the error covariance matrices for the states predicted by previous information.

(2) Comparison between the proposed GEKF-UI with existing EKF approaches
It is noted from Eqs. (21)–(28) that the solutions of the proposed GEKF-UI collapse to (i) the solutions of the classical EKF [1], when all the inputs are known, i.e., $\mathbf{B}^*_{k|k} = \mathbf{0}$ and $\mathbf{D}^*_{k+1|k} = \mathbf{0}$; (ii) the solutions of EKF-UI [15] when all the unknown inputs are presented in the observation equations, i.e., $\mathbf{D}^*_{k+1|k} = \bar{\mathbf{D}}^*_{k+1|k}$ where $\bar{\mathbf{D}}^*_{k+1|k}$ has full column rank; and (iii) the solutions of EKF-UI-WDF [17], when none of unknown inputs exists in the observation equations, i.e., $\mathbf{D}^*_{k+1|k} = \mathbf{0}$ while $\mathbf{B}^*_{k|k}$ has full column rank. Therefore, the proposed GEKF-UI forms a general EKF approach covering different scenarios of unknown input. A detailed comparison of the above EKF formulations is shown in Table 1.

(3) Advantages of the proposed GEKF-UI over the combined traditional EKF and LSE methods
Methods can be found combining the traditional EKF and LSE method [13,16,17] to estimate the states and the unknown inputs sequentially. They form a special case of the proposed GEKF-UI with the final solutions similar to that of EKF-UI. It was claimed by Lei et al. [16] that the formulation of the sequential method is more concise than that of EKF-UI. More discussion on the advantages of the proposed GEKF-UI approach compared to the traditional EKF and LSE methods is given below. The latter method is referred to as the combined method:
   (i) The proposed GEKF-UI is derived by minimizing an objective function of Eq. (18), which is the same as that for the traditional EKF (e.g., [1]). The mathematical basis of the GEKF-UI is more rigorous based on global optimization instead of local optimization in the combined method. The combined methods [13,16] are obtained by embedding the LSE solution for unknown inputs into the traditional EKF approach at each time instance. The resulting solutions are locally optimal but the global optimality is not guaranteed. On the other hand, the proposed GEKF-UI is obtained by minimizing the global objective function with respect to the states and all the unknown inputs with a global optimality implication.
   (ii) It is noted from Eq. (28) that the updated solution of the state error covariance matrix $\mathbf{P}_{\mathbf{Z},k|k}$ is not only related to the previous state error covariance matrix $\mathbf{P}_{\mathbf{Z},k|k-1}$, but also to the input error covariance matrix $\mathbf{S}_k$. This is consistent with the common belief that the estimation errors of the current states are affected by the errors of both the previous states and the current unknown inputs. The combined methods [13,16] just directly adopt $\mathbf{P}_{\mathbf{Z},k|k}$ of the traditional EKF without considering the error introduced by the estimated unknown inputs. The updated solution of the state error covariance matrix is the same as that for the traditional EKF incorporating no new information on the current unknown inputs. The omission of influence of the unknown inputs may result in unexpected instability or inaccuracy of the estimation.
   (iii) Lastly, the proposed GEKF-UI can provide the state and unknown input estimation for the general case when the unknown inputs are partially presented in the observation equations (i.e. $0 < rank(\mathbf{D}^*_{k+1|k}) < r$). To the best of the authors' knowledge, this scenario of measurement cannot be handled by other existing EKF-based methods.

4. Numerical examples

To demonstrate the effectiveness and accuracy of the proposed GEKF-UI approach, simulation results from (a) a 3-DOFs linear damped shear building; (b) the Phase I ASCE benchmark for structural health monitoring; and (c) a 2-storey plane frame with abrupt stiffness reduction (damage) will be presented herein. All the "measured" responses are simulated from the theoretically computed quantities superimposed with 5% RMS white noise in the following studies except otherwise stated.

To quantitatively evaluate the performance of the proposed GEKF-UI in above numerical examples, the error of estimation is

Table 1
Comparison of the proposed GEKF-UI with existing EKF approaches.

EKF methods	Model and measurement equations	Measurement update (correction)	Time update (prediction)
Traditional EKF [1]	(1) Model equation $\dot{\mathbf{Z}}(t) = \mathbf{g}(\mathbf{Z}, \mathbf{f}, t) + \mathbf{w}(t)$ (2) Measurement equation $\mathbf{y}_{k+1} = \mathbf{h}(\mathbf{Z}_{k+1}, \mathbf{f}_{k+1}, k+1) + \mathbf{v}_{k+1}$ (3) Condition for the estimation of unknown inputs Not available	(1) Estimation for states $\hat{\mathbf{Z}}_{k+1\mid k+1} = \hat{\mathbf{Z}}_{k+1\mid k} + \mathbf{K}_{\mathbf{Z},k+1}[\mathbf{y}_{k+1} - \mathbf{h}(\hat{\mathbf{Z}}_{k+1\mid k}, \mathbf{f}_k, k+1)]$ (2) Estimation for unknown inputs in the observation equations Not available (3) Estimation for Unknown inputs only in the model equations Not available (4) Gain matrix for state estimation $\mathbf{K}_{\mathbf{Z},k+1} = \mathbf{P}_{\mathbf{Z},k+1\mid k}\mathbf{H}^{\mathrm{T}}_{k+1\mid k}[\mathbf{H}_{k+1\mid k}\mathbf{P}_{\mathbf{Z},k+1\mid k}\mathbf{H}^{\mathrm{T}}_{k+1\mid k} + \mathbf{R}_{k+1}]^{-1}$ (5) Correction of error covariance matrix of states $\mathbf{P}_{\mathbf{Z},k\mid k} = (\mathbf{I}_n - \mathbf{K}_{\mathbf{Z},k}\mathbf{H}_{k\mid k-1})\mathbf{P}_{\mathbf{Z},k\mid k-1}$ (6) Error covariance matrix of unknown inputs Not available	(1) Prediction for states $\hat{\mathbf{Z}}_{k+1\mid k} = \hat{\mathbf{Z}}_{k\mid k} + \int_{k\Delta t}^{(k+1)\Delta t} \mathbf{g}(\hat{\mathbf{Z}}_{t\mid k}, \mathbf{f}, t)dt$ (2) Prediction of error covariance matrix of states $\mathbf{P}_{\mathbf{Z},k+1\mid k} = \mathbf{\Phi}_{k+1,k}\mathbf{P}_{\mathbf{Z},k\mid k}\mathbf{\Phi}^{\mathrm{T}}_{k+1,k} + \mathbf{Q}_{k+1}$
EKF-UI [15]	(1) Model equation $\dot{\mathbf{Z}}(t) = \mathbf{g}(\mathbf{Z}, \mathbf{f}, \mathbf{f}^*, t) + \mathbf{w}(t)$ (2) Measurement equation $\mathbf{y}_{k+1} = \mathbf{h}(\mathbf{Z}_{k+1}, \mathbf{f}_{k+1}, \mathbf{f}^*_{k+1}, k+1) + \mathbf{v}_{k+1}$ (3) Condition for the estimation of unknown inputs $\mathbf{D}^*_{k+1\mid k}$ has full column rank.	(1) Estimation for states $\hat{\mathbf{Z}}_{k+1\mid k+1} = \hat{\mathbf{Z}}_{k+1\mid k} + \mathbf{K}_{\mathbf{Z},k+1}[\mathbf{y}_{k+1} - \mathbf{h}(\hat{\mathbf{Z}}_{k+1\mid k}, \mathbf{f}_k, \hat{\hat{\mathbf{f}}}^*_{k+1\mid k+1}, k+1)]$ (2) Estimation for unknown inputs in the observation equations $\hat{\mathbf{f}}^*_{k+1\mid k+1} = \mathbf{S}_{k+1}\mathbf{D}^{*T}_{k+1\mid k}\mathbf{R}^{-1}_{k+1}(\mathbf{I}_l - \mathbf{H}_{k+1\mid k}\mathbf{K}_{\mathbf{Z},k+1})(\mathbf{y}_{k+1} - \hat{\mathbf{h}}_{k+1\mid k} + \mathbf{D}^*_{k+1\mid k}\hat{\mathbf{f}}^*_{k\mid k})$ (3) Estimation for Unknown inputs only in the model equations Not available (4) Gain matrix for state estimation $\mathbf{K}_{\mathbf{Z},k+1} = \mathbf{P}_{\mathbf{Z},k+1\mid k}\mathbf{H}^{\mathrm{T}}_{k+1\mid k}[\mathbf{H}_{k+1\mid k}\mathbf{P}_{\mathbf{Z},k+1\mid k}\mathbf{H}^{\mathrm{T}}_{k+1\mid k} + \mathbf{R}_{k+1}]^{-1}$ (5) Correction of error covariance matrix of states $\mathbf{P}_{\mathbf{Z},k\mid k} = (\mathbf{I}_n - \mathbf{K}_{\mathbf{Z},k}\mathbf{H}_{k\mid k-1})[\mathbf{P}_{\mathbf{Z},k\mid k-1} +$ $(\mathbf{H}^{\mathrm{T}}_{k\mid k-1}\mathbf{R}^{-1}_k\mathbf{D}^*_{k\mid k-1})\mathbf{S}_k(\mathbf{H}^{\mathrm{T}}_{k\mid k-1}\mathbf{R}^{-1}_k\mathbf{D}^*_{k\mid k-1})^{\mathrm{T}}(\mathbf{I}_n - \mathbf{K}_{\mathbf{Z},k}\mathbf{H}_{k\mid k-1})^{\mathrm{T}}]$ (6) Error covariance matrix of unknown inputs $\mathbf{S}_{k+1} = \left[\mathbf{D}^{*\mathrm{T}}_{k+1\mid k}\mathbf{R}^{-1}_{k+1}(\mathbf{I}_l - \mathbf{H}_{k+1\mid k}\mathbf{K}_{\mathbf{Z},k+1})\mathbf{D}^*_{k+1\mid k}\right]^{-1}$	(1) Prediction for states $\hat{\mathbf{Z}}_{k+1\mid k} = \hat{\mathbf{Z}}_{k\mid k} + \int_{k\Delta t}^{(k+1)\Delta t} \mathbf{g}(\hat{\mathbf{Z}}_{t\mid k}, \mathbf{f}, \hat{\mathbf{f}}^*_{t\mid k}, t)dt$ (2) Prediction of error covariance matrix of states $\mathbf{P}_{\mathbf{Z},k+1\mid k} = \mathbf{\Phi}_{k+1,k}\mathbf{P}_{\mathbf{Z},k\mid k}\mathbf{\Phi}^{\mathrm{T}}_{k+1,k} + \mathbf{Q}_{k+1}$
EKF-UI-WDF [18]	(1) Model equation $\dot{\mathbf{Z}}(t) = \mathbf{g}(\mathbf{Z}, \mathbf{f}, \mathbf{f}^*, t) + \mathbf{w}(t)$ (2) Measurement equation $\mathbf{y}_{k+1} = \mathbf{h}(\mathbf{Z}_{k+1}, \mathbf{f}_{k+1}, \mathbf{f}^*_{k+1}, k+1) + \mathbf{v}_{k+1}$ (3) Condition for the estimation of unknown inputs $\mathbf{B}^*_{k\mid k}$ has full column rank and $\mathbf{D}^*_{k+1\mid k} = \mathbf{0}$	(1) Estimation for states $\hat{\mathbf{Z}}_{k+1\mid k+1} = \hat{\mathbf{Z}}_{k+1\mid k} + \mathbf{K}_{\mathbf{Z},k+1}[\mathbf{y}_{k+1} - \mathbf{h}(\hat{\mathbf{Z}}_{k+1\mid k}, \mathbf{f}_k, \hat{\hat{\mathbf{f}}}^*_{k+1\mid k+1}, k+1)]$ (2) Estimation for unknown inputs in the observation equations Not available (3) Estimation for Unknown inputs only in the model equations $\hat{\mathbf{f}}^*_{k\mid k+1} = \mathbf{S}_{k+1}\mathbf{B}^{*\mathrm{T}}_k\mathbf{H}^{\mathrm{T}}_{k+1\mid k}(\mathbf{I}_l - \mathbf{H}_{k+1\mid k}\mathbf{K}_{\mathbf{Z},k+1})$ $(\mathbf{y}_{k+1} - \hat{\mathbf{h}}_{k+1\mid k} + \mathbf{H}_{k+1\mid k}\mathbf{B}^*_k\hat{\mathbf{f}}^*_{k-1\mid k})$ (4) Gain matrix for state estimation $\mathbf{K}_{\mathbf{Z},k+1} = \mathbf{P}_{\mathbf{Z},k+1\mid k}\mathbf{H}^{\mathrm{T}}_{k+1\mid k}[\mathbf{H}_{k+1\mid k}\mathbf{P}_{\mathbf{Z},k+1\mid k}\mathbf{H}^{\mathrm{T}}_{k+1\mid k} + \mathbf{R}_{k+1}]^{-1}$ (5) Correction of error covariance matrix of states $\mathbf{P}_{\mathbf{Z},k\mid k} = (\mathbf{I}_n - \mathbf{K}_{\mathbf{Z},k}\mathbf{H}_{k\mid k-1})[\mathbf{P}_{\mathbf{Z},k\mid k-1} + \tilde{\mathbf{B}}^*_{k-1}\mathbf{S}_k\tilde{\mathbf{B}}^{*\mathrm{T}}_{k-1}(\mathbf{I}_n - \mathbf{K}_{\mathbf{Z},k}\mathbf{H}_{k\mid k-1})^{\mathrm{T}}]$ (6) Error covariance matrix of unknown inputs $\mathbf{S}_{k+1} = \left[\mathbf{B}^{*\mathrm{T}}_k\mathbf{H}^{\mathrm{T}}_{k+1\mid k}\mathbf{R}^{-1}_{k+1}(\mathbf{I}_l - \mathbf{H}_{k+1\mid k}\mathbf{K}_{\mathbf{Z},k+1})\mathbf{H}_{k+1\mid k}\mathbf{B}^*_k\right]^{-1}$	(1) Prediction for states $\hat{\mathbf{Z}}_{k+1\mid k} = \hat{\mathbf{Z}}_{k\mid k} + \int_{k\Delta t}^{(k+1)\Delta t} \mathbf{g}(\hat{\mathbf{Z}}_{t\mid k}, \mathbf{f}, \hat{\mathbf{f}}^*_{t\mid k}, t)dt$ (2) Prediction of error covariance matrix of states $\mathbf{P}_{\mathbf{Z},k+1\mid k} = \mathbf{\Phi}_{k+1,k}\mathbf{P}_{\mathbf{Z},k\mid k}\mathbf{\Phi}^{\mathrm{T}}_{k+1,k} + \mathbf{Q}_{k+1}$

GEKF-UI	(1) Model equation $\dot{\mathbf{Z}}(t)=\mathbf{g}(\mathbf{Z},\mathbf{f},\mathbf{f}^*,t)+\mathbf{w}(t)$ (2) Measurement equation $\mathbf{y}_{k+1}=\mathbf{h}(\mathbf{Z}_{k+1},\mathbf{f}_{k+1},\mathbf{f}^*_{k+1},k+1)+\mathbf{v}_{k+1}$ (3) Condition for the estimation of unknown inputs $\mathbf{D}^*_{k+1\mid k}=\begin{bmatrix}\mathbf{0}_{l\times\bar{r}} & \bar{\mathbf{D}}^*_{k+1\mid k}\end{bmatrix}$ and $\mathbf{B}^*_k=\begin{bmatrix}\tilde{\mathbf{B}}^*_k & \bar{\mathbf{B}}^*_k\end{bmatrix}$ where $\bar{\mathbf{D}}^*_{k+1\mid k}$ has full column rank $\bar{r}$ and $\tilde{\mathbf{B}}^*_k$ has full column rank $\tilde{r}$.	(1) Estimation for states $\hat{\mathbf{Z}}_{k+1\mid k+1}=\hat{\mathbf{Z}}_{k+1\mid k}+\mathbf{K}_{\mathbf{Z},k+1}[\mathbf{y}_{k+1}-\mathbf{h}(\hat{\mathbf{Z}}_{k+1\mid k},\mathbf{f}_k,\hat{\bar{\mathbf{f}}}^*_{k+1\mid k+1},k+1)]$ (2) Estimation for unknown inputs in the observation equations $\hat{\bar{\mathbf{f}}}^*_{k+1\mid k+1}=\mathbf{S}_{k+1,22}\bar{\mathbf{D}}^{*\mathrm{T}}_{k+1\mid k}\mathbf{R}^{-1}_{k+1}(\mathbf{I}_l-\mathbf{H}_{k+1\mid k}\mathbf{K}_{\mathbf{Z},k+1})(\mathbf{y}_{k+1}-\hat{\mathbf{h}}_{k+1\mid k}+\bar{\mathbf{D}}^*_{k+1\mid k}\hat{\bar{\mathbf{f}}}^*_{k\mid k})$ $+\mathbf{S}_{k+1,12}\tilde{\mathbf{B}}^{*\mathrm{T}}_k\mathbf{H}^{\mathrm{T}}_{k+1\mid k}(\mathbf{I}_l-\mathbf{H}_{k+1\mid k}\mathbf{K}_{\mathbf{Z},k+1})(\mathbf{y}_{k+1}-\hat{\mathbf{h}}_{k+1\mid k}+\mathbf{H}_{k+1\mid k}\tilde{\mathbf{B}}^*_k\hat{\tilde{\mathbf{f}}}^*_{k-1\mid k})$ (3) Estimation for Unknown inputs only in the model equations $\hat{\tilde{\mathbf{f}}}^*_{k\mid k+1}=\mathbf{S}_{k+1,12}\bar{\mathbf{D}}^{*\mathrm{T}}_{k+1\mid k}\mathbf{R}^{-1}_{k+1}(\mathbf{I}_l-\mathbf{H}_{k+1\mid k}\mathbf{K}_{\mathbf{Z},k+1})(\mathbf{y}_{k+1}-\hat{\mathbf{h}}_{k+1\mid k}+\bar{\mathbf{D}}^*_{k+1\mid k}\hat{\bar{\mathbf{f}}}^*_{k\mid k})$ $+\mathbf{S}_{k+1,11}\tilde{\mathbf{B}}^{*\mathrm{T}}_k\mathbf{H}^{\mathrm{T}}_{k+1\mid k}(\mathbf{I}_l-\mathbf{H}_{k+1\mid k}\mathbf{K}_{\mathbf{Z},k+1})(\mathbf{y}_{k+1}-\hat{\mathbf{h}}_{k+1\mid k}+\mathbf{H}_{k+1\mid k}\tilde{\mathbf{B}}^*_k\hat{\tilde{\mathbf{f}}}^*_{k-1\mid k})$ (4) Gain matrix for state estimation $\mathbf{K}_{\mathbf{Z},k+1}=\mathbf{P}_{\mathbf{Z},k+1\mid k}\mathbf{H}^{\mathrm{T}}_{k+1\mid k}[\mathbf{H}_{k+1\mid k}\mathbf{P}_{\mathbf{Z},k+1\mid k}\mathbf{H}^{\mathrm{T}}_{k+1\mid k}+\mathbf{R}_{k+1}]^{-1}$ (5) Correction of error covariance matrix of states $\mathbf{P}_{\mathbf{Z},k\mid k}=(\mathbf{I}_n-\mathbf{K}_{\mathbf{Z},k}\mathbf{H}_{k\mid k-1})\{\mathbf{P}_{\mathbf{Z},k\mid k-1}+$ $\begin{bmatrix}\tilde{\mathbf{B}}^*_{k-1} & \mathbf{H}^{\mathrm{T}}_{k\mid k-1}\mathbf{R}^{-1}_k\bar{\mathbf{D}}^*_{k\mid k-1}\end{bmatrix}\mathbf{S}_k\begin{bmatrix}\tilde{\mathbf{B}}^*_{k-1} & \mathbf{H}^{\mathrm{T}}_{k\mid k-1}\mathbf{R}^{-1}_k\bar{\mathbf{D}}^*_{k\mid k-1}\end{bmatrix}^{\mathrm{T}}(\mathbf{I}_n-\mathbf{K}_{\mathbf{Z},k}\mathbf{H}_{k\mid k-1})^{\mathrm{T}}\}$ (6) Error covariance matrix of unknown inputs $\mathbf{S}_{k+1}=\left[\begin{array}{c\|c}\tilde{\mathbf{B}}^{*\mathrm{T}}_k\mathbf{H}^{\mathrm{T}}_{k+1\mid k}\mathbf{R}^{-1}_{k+1}(\mathbf{I}_l-\mathbf{H}_{k+1\mid k}\mathbf{K}_{\mathbf{Z},k+1})\mathbf{H}_{k+1\mid k}\tilde{\mathbf{B}}^*_k & \tilde{\mathbf{B}}^{*\mathrm{T}}_k\mathbf{H}^{\mathrm{T}}_{k+1\mid k}\mathbf{R}^{-1}_{k+1}(\mathbf{I}_l-\mathbf{H}_{k+1\mid k}\mathbf{K}_{\mathbf{Z},k+1})\bar{\mathbf{D}}^*_{k+1\mid k} \\ \hline \bar{\mathbf{D}}^{*\mathrm{T}}_{k+1\mid k}\mathbf{R}^{-1}_{k+1}(\mathbf{I}_l-\mathbf{H}_{k+1\mid k}\mathbf{K}_{\mathbf{Z},k+1})\mathbf{H}_{k+1\mid k}\tilde{\mathbf{B}}^*_k & \bar{\mathbf{D}}^{*\mathrm{T}}_{k+1\mid k}\mathbf{R}^{-1}_{k+1}(\mathbf{I}_l-\mathbf{H}_{k+1\mid k}\mathbf{K}_{\mathbf{Z},k+1})\bar{\mathbf{D}}^*_{k+1\mid k}\end{array}\right]^{-1}$	(1) Prediction for states $\hat{\mathbf{Z}}_{k+1\mid k}=\hat{\mathbf{Z}}_{k\mid k}+\int_{k\Delta t}^{(k+1)\Delta t}\mathbf{g}(\hat{\mathbf{Z}}_{t\mid k},\mathbf{f},\hat{\mathbf{f}}^*_{t\mid k},t)dt$ (2) Prediction of error covariance matrix of states $\mathbf{P}_{\mathbf{Z},k+1\mid k}=\mathbf{\Phi}_{k+1,k}\mathbf{P}_{\mathbf{Z},k\mid k}\mathbf{\Phi}^{\mathrm{T}}_{k+1,k}+\mathbf{Q}_{k+1}$

quantified with the mean normalized 2-norm of the estimation error time history ε_i [21] as

$$\varepsilon_i=\frac{1}{N}\sum_{j=1}^{N}\left[\frac{\sum_{k=1}^{n}[z_i(k)-\hat{z}_{i,j}(k)]^2}{\sum_{k=1}^{n}[z_i(k)]^2}\right]^{1/2} \tag{29}$$

where $z_i(k)$ is the ith theoretical states (inputs) and $\hat{z}_{i,j}(k)$ is the corresponding estimate from jth random simulation; N is the number of random simulations and n is the number of time steps in simulation.

In the formulation of the proposed GEKF-UI as described in Eqs. (21)–(28), the matrix $\mathbf{R}_k$ in Eq. (28) is theoretically defined as the covariance matrix of the zero mean Gaussian measurement noise vector $\mathbf{v}_k, \mathrm{E}[\mathbf{v}_k\mathbf{v}_k^T]=\mathbf{R}_k$ to quantify the measurement uncertainties. It has significant effect on the performance of the proposed GEKF-UI. However, the real unknown vector $\mathbf{v}_k$ considers errors from both measurement noise and modeling error due to the linearization of the nonlinear functions, and thus it is not a zero mean Gaussian vector any more. As a result, matrix $\mathbf{R}_k$ was empirically chosen as a constant positive diagonal matrix, i.e., $\mathbf{R}_k=\mathbf{R}=a\mathbf{I}_l$ (usually with $0<a\leqslant 1$) in the literature [1–5], rather than being obtained directly from its theoretical definition. In this paper, an iterative procedure is applied to determine $\mathbf{R}_k$ by (i) tentatively choosing a $\mathbf{R}_k=a_0\mathbf{I}_l$ which leads to convergence of the proposed GEKF-UI; and (ii) smoothly adjusting the value of a from a_0 to minimize the mean normalized 2-norm of the estimation error of the structural parameters obtained from Eq. (29), until a satisfactory $\mathbf{R}_k=a\mathbf{I}_l$ is obtained. The major limitation of this method is that it is a trial and error procedure where the optimality of the $\mathbf{R}_k$ obtained is not guaranteed. This is a challenging problem for further research associated with the EKF approaches. Similarly, the matrix $\mathbf{Q}_{k+1}$ in Eq. (27) is also chosen as $\mathbf{Q}_{k+1}=\mathbf{Q}=b\mathbf{I}_{2m+n}$ (usually with $0<b\leqslant 1$) and the associated discussion is omitted herein.

4.1. DOFs linear structure

Considering a 3-storey (3-DOFs) linear damped shear building subjected to El Centro seismic excitation with acceleration $\ddot{x}_0$. The equations of motion are given by

$$\begin{aligned}
m_1(\ddot{x}_1+\ddot{x}_0)&=-c_1\dot{x}_1+c_2(\dot{x}_2-\dot{x}_1)-k_1x_1+k_2(x_2-x_1)\\
m_2(\ddot{x}_2+\ddot{x}_0)&=-c_2(\dot{x}_2-\dot{x}_1)+c_3(\dot{x}_3-\dot{x}_2)-k_2(x_2-x_1)+k_3(x_3-x_2)\\
m_3(\ddot{x}_3+\ddot{x}_0)&=-c_3(\dot{x}_3-\dot{x}_2)-k_3(x_3-x_2)
\end{aligned} \tag{30}$$

where x_i is the relative displacement of the ith storey with respect to the base. $m_1=m_2=m_3=1000$ kg, $c_1=c_2=c_3=0.6$ kN s/m, $k_1=120$ kN/m, $k_2=120$ kN/m, $k_3=60$ kN/m. The natural frequencies of structure are $\omega_1=0.73$ Hz, $\omega_2=1.74$ Hz, and $\omega_3=2.93$ Hz.

The seismic excitation $\ddot{\mathbf{x}}_0$ with a peak ground acceleration of 1.0 g is applied to the structure. The "measured" inter-storey drifts of the 1st, 2nd and 3rd stories, i.e., $d_1=x_1, d_2=x_2-x_1$ and $d_3=x_3-x_2$ are simulated by adding to the theoretically computed responses with 5% RMS white noise. The sampling frequency is 500 Hz.

Unknown quantities to be identified are: (i) the state variables (the displacements and velocities of three stories); (ii) the structural parameters k_1,k_2,k_3,c_1,c_2 and c_3; and (iii) the unknown earthquake ground acceleration $\ddot{x}_0(t)$. Given the measurement vector $\mathbf{y}=[d_1d_2d_3]^{\mathrm{T}}$, the coefficient matrix $\mathbf{D}^*_{k+1\mid k}=[0\quad 0\quad 0]^{\mathrm{T}}$ is obtained.

The initial values of the state variables are assumed to be zeros, and the initial values for k_1,k_2,k_3,c_1,c_2 and c_3 are taken as $c_{1,0}=c_{2,0}=c_{3,0}=0.4$ kN-s/m, and $k_{1,0}=k_{2,0}=100$ kN/m,

$k_{3,0} = 50$ kN/m, and the diagonal elements of the initial error covariance matrix of the extended state vector $\mathbf{P}_{Z,0|0}$ are $[1,1,1,1,1,1,10^8,10^8,10^8,10^8,10^8,10^8]^T$. The covariance matrices of both the measurement noise vector $\mathbf{v}(t)$ and the system noise vector $\mathbf{w}(t)$ are chosen to be $\mathbf{R} = 0.1\mathbf{I}_3$ and $\mathbf{Q} = 10^{-2}\mathbf{I}_{12}$, respectively, where the subscript denotes the dimension of the unity square matrix.

The proposed GEKF-UI approach is conducted and the identified parameters (k_i and c_i for $i = 1, 2, 3$) are presented in Fig. 1. All the curves are overlapping except in the first two seconds. The identified earthquake ground acceleration $f_e^* = \ddot{x}_0$ is presented in Fig. 2. The estimation error $\delta f_e^* = \hat{f}^*_{e,k+1|k+1} - f_e^*$ given in Fig. 2 is small except in the first 0.2 s. These results show that the proposed method is accurate.

The velocity response and damping parameter of the 2nd storey estimated by GEKF-UI and traditional EKF are presented in Fig. 3. Large error is noted in the parameter estimates of the traditional EKF where the unknown input is taken as null.

Using measurements with time segment of 5 s (between $t = 3$ and 8 s) with a sampling intervals $\Delta t = 0.002$ s from 100 random simulations (i.e., $N = 100$ and $n = 2501$) and under 5% and 10% RMS noise, the mean estimation errors (Eq. (29)) with traditional EKF and GEKF-UI are presented in Table 2. The errors are noted very small compared to those from the traditional EKF.

4.2. ASCE Phase I benchmark structure

This benchmark structure consists of four storeys three dimensional steel frame. Two white noise excitations are applied at the 2nd and 4th floors in the weak direction of the building. Due to symmetry, the structure is reduced to a 4-DOFs shear-beam model. The masses of floors are $m_1 = 3.4524$ tons, $m_2 = m_3 = 2.6524$ tons and $m_4 = 1.8099$ tons, and the stiffness for all stories are identical, with $k_1 = k_2 = k_3 = k_4 = 67.9$ MN/m. The damping matrix $\mathbf{C}$ is assumed to be proportional to the stiffness matrix $\mathbf{K}$, i.e. $\mathbf{C} = 1 \times 10^{-4}\,\mathbf{K}$. Hence, the damping coefficient of each storey unit is $c_1 = c_2 = c_3 = c_4 = 6.79$ kN s/m. The natural frequencies ω_i are: $\omega_1 = 9.41$ Hz, $\omega_2 = 25.54$ Hz, $\omega_3 = 38.66$ Hz, and $\omega_4 = 48.01$ Hz.

To show the capability of the proposed GEKF-UI, only the absolute accelerations of the 1st, 3rd and 4th floors are measured. As a result, the measured output vector is $\mathbf{y} = [\ddot{x}_1\ddot{x}_3\ddot{x}_4]^T$ and the coefficient matrix corresponding to the unknown input vector $\mathbf{f}_e^* = [f_2^* \quad f_4^*]^T$ is $\mathbf{D}^*_{k+1|k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$. Matrix $\mathbf{D}^*_{k+1|k} = [\mathbf{0}_{3\times1} \;\vdots\; \bar{\mathbf{D}}^*_{k+1|k}]$ is partitioned with $\bar{\mathbf{D}}^*_{k+1|k} = [0 \quad 0 \quad 1]^T$ according to Eq. (17).

The white noise excitation at the 2nd and 4th floors are not measured. The sampling frequency is 1000 Hz. Other than the state variables, the unknown quantities to be identified include the structural parameters c_i and k_i (for $i = 1, 2, 3, 4$) and the white noise excitations f_2 and f_4 at the 2nd and 4th floors.

The initial values for the state variables are assumed zero and the initial guesses for c_1, c_2, c_3, c_4, and k_1, k_2, k_3, k_4 are respectively: $c_{1,0} = c_{2,0} = c_{3,0} = c_{4,0} = 4.0$ kN-s/m and $k_{1,0} = k_{2,0} = k_{3,0} = k_{4,0} = 40.0$ MN/m. The initial error covariance matrix of the extended state vector is taken to be the diagonal matrix $\mathbf{P}_{Z,0|0}$ with diagonal elements [1,1,1,1,1,1,1,1]. The covariance matrices of both the measurement noise vector $\mathbf{v}(t)$ and the model noise vector $\mathbf{w}(t)$ are chosen to be $\mathbf{R} = 0.1\mathbf{I}_3$ and $\mathbf{Q} = 10^{-3}\mathbf{I}_{16}$, respectively.

Based on the proposed GEKF-UI, the estimated parameters for a time duration of 10 s are presented in Fig. 4. The estimated stiffness parameters are almost identical while the estimated damping curves are almost overlapping. The errors in the estimated structural parameters are noted small while results (not shown) from the traditional EKF diverge without convergence.

The identified unknown white noise excitations f_2 and f_4 within the time duration between 3 and 3.1 s are presented in Fig. 5. The estimation errors $\delta f_2 = \hat{f}_{2,k|k+1} - f_2$ and $\delta f_4 = \hat{f}_{4,k+1|k+1} - f_4$ are also shown in the figure. The errors of estimation calculated from Eq. (29) are given in Table 3 for responses with 5% and 10% noise.

4.3. Time-varying estimation with a 2-storey steel plane frame

Consider a 2-storey plane steel frame [22] with 6 members, 4 nodes and 12 DOFs in the finite element model. All members are of the same uniform cross-section. White noise excitations f_1 and f_2 are applied on Nodes 1 and 2 horizontally and vertically, respectively. The dynamic effect of the second force is small and it is specially selected for illustration of the effectiveness of the proposed method. The equation of motion of the plane steel frame is given as follows

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \boldsymbol{\eta}\mathbf{f}^*(t) \tag{31}$$

where $\mathbf{x}(t) = [z_{1x}, z_{1y}, z_{1\theta}, \ldots, z_{4x}, z_{4y}, z_{4\theta}]^T = [x_1, x_2, \ldots, x_{12}]^T$ and z_{jx}, z_{jy} and $z_{j\theta}$ are the translational displacements relative to the ground, and rotational displacements of the jth node ($j = 1, 2, 3, 4$) respectively; $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are (12×12) global mass, damping and stiffness matrices for the finite element structure, respectively; $\mathbf{f}^* = [f_1^* \quad f_2^*]^T$ is the external excitation vector; and $\boldsymbol{\eta} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$ is (12×2) mapping matrix for the excitation.

Consistent mass matrix is adopted and the equivalent stiffness parameter $k_i = \frac{E_i I_i}{L_i}$ for each member is adopted for the estimation, where E_i, I_i, and A_i are the Young's modulus, moment of inertia, and area of the cross-section of the ith element, respectively. The stiffness matrix is represented as the summation of the elemental matrices as

$$\mathbf{K} = \sum_{i=1}^{6} \tilde{\mathbf{K}}_i = \sum_{i=1}^{6} k_i \mathbf{S}_i \tag{32}$$

where $\mathbf{S}_i$ is the (12×12) mapping matrix of the ith element.

Rayleigh damping is assumed and the damping matrix $\mathbf{C}$ is expressed as

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{33}$$

where α is the mass-proportional damping coefficient and β is the stiffness-proportional damping coefficient.

The physical and material parameters of members are: $A_i = 1.34 \times 10^{-2}$ m^2, $I_i = 4.87 \times 10^{-4}$ m^4 ($i = 1, 2, \ldots, 6$), mass per unit length of member, $\bar{m}_i = 10.71$ kg/m ($i = 1, 2, \ldots, 6$), $k_i = 10.65$ MN m ($i = 1, 2$), $k_i = 26.61$ MN-m ($i = 3, 4, 5, 6$), $\alpha = 1.94$ s^{-1}, and $\beta = 3.19 \times 10^{-4}$ s. The first natural frequency and the corresponding damping ratio of this frame are 33.08 Hz and 3.79%, respectively. The sampling frequency is 1000 Hz.

Unlike the previous two numerical examples with constant parameters, a damage pattern is considered with the stiffness parameter in Element 6 in the first storey changes abruptly at $t = 3$ s, i.e., k_6 is reduced from 26,611 kN-m to 22,619 kN-m (15% reduction). Other parameters are constant throughout the time duration studied. Besides the state variables, the unknown parameters to be identified are: k_i ($i = 1, 2, \ldots, 6$), α and β, and the excitations f_1 and f_2.

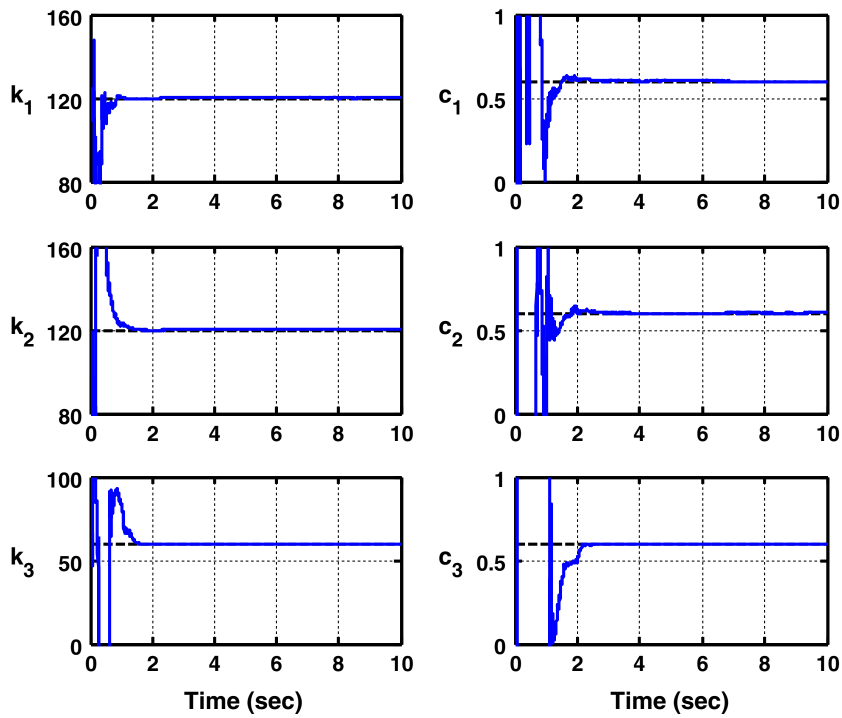

Fig. 1. Estimated parameters for a 3-storey shear building. (Unit of k_i in kN/m and c_i in kN-s/m ($i = 1, 2, 3$).) (—— estimated; ------ theoretical).

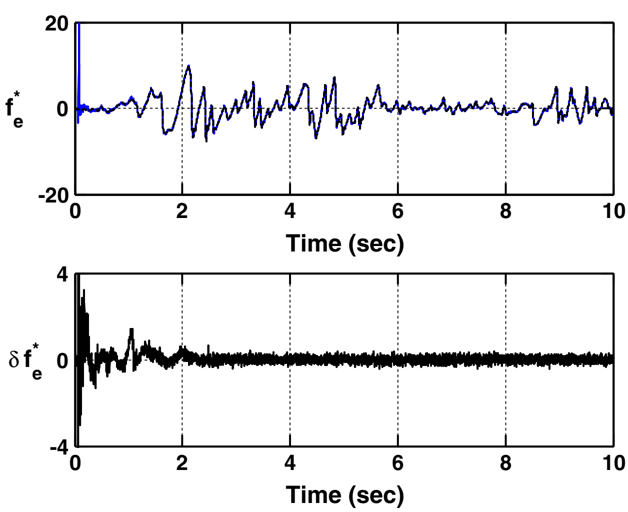

Fig. 2. Estimated unknown input $f_e^* = \ddot{x}_0$ and error of estimation δf_e^* for a 3-storey shear building. (Unit of f_e^* and δf_e^* in m/s^2.) (—— estimated; ------ theoretical).

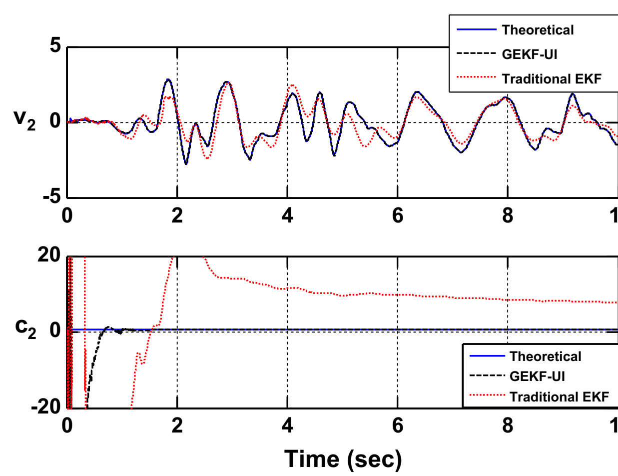

Fig. 3. Comparison of estimates of v_2 and c_2 or a 3-storey shear building. (Unit of v_2 in m/s and c_2 in kN-s/m.)

To track the abrupt reduction of k_6 on-line, an adaptive technique proposed by Yang et al. [14] is used by replacing $\mathbf{P}_{\mathbf{Z},k+1|k}$ in Eq. (27) with the following equation,

$$\mathbf{P}_{\mathbf{Z},k+1|k} = \boldsymbol{\Lambda}_{k+1}[\boldsymbol{\Phi}_{k+1,k}\mathbf{P}_{\mathbf{Z},k|k}\boldsymbol{\Phi}_{k+1,k}^{\mathrm{T}}]\boldsymbol{\Lambda}_{k+1}^{\mathrm{T}} + \mathbf{Q}_{k+1} \tag{34}$$

where $\boldsymbol{\Lambda}_{k+1}$ is an $(32{\times}32)$ adaptive factor matrix. The matrices $\boldsymbol{\Phi}_{k+1,k}$ and $\mathbf{Q}_{k+1}$ have been defined in Section 2.

The observation vector $\mathbf{Y} = [\ddot{x}_2, \ddot{x}_3, \ddot{x}_4, \ddot{x}_5, \ddot{x}_6, \ddot{x}_7, \ddot{x}_8, \ddot{x}_9, \ddot{x}_{10}, \ddot{x}_{11}]^{\mathrm{T}}$ has 10 elements. The subscripts denote the *dofs*. The acceleration in the horizontal direction of 1*st* node where the unknown input f_1^* is acting, i.e. $\ddot{x}_1$ is not measured. This implies $0 < rank(\mathbf{D}_{k+1|k}^*) = 1 < 2$ (full column rank). The coefficient matrix corresponding to the unknown input vector $\mathbf{f}_e^* = [f_1^* \quad f_2^*]^{\mathrm{T}}$ is $\mathbf{D}_{k+1|k}^* = [\,\mathbf{0}_{10\times1} \;\vdots\; \bar{\mathbf{D}}_{k+1|k}^*\,]$ and $\bar{\mathbf{D}}_{k+1|k}^* = [0 \;\; 0 \;\; 0 \;\; -1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^{\mathrm{T}}$ according to Eq. (17).

The initial values for the state variables are assumed zero and the initial guesses for $k_i(i = 1,\ 2,\ 3,\ 4,\ 5,\ 6\ )$, α and β are $k_{1,0} = k_{2,0} = 8.00$ MN/m, $k_{3,0} = k_{4,0} = k_{5,0} = k_{6,0} = 20.00$ MN/m, $\alpha_0 = 1$ m^2/s and $\beta_0 = 2 \times 10^{-4}$ s, respectively. The (32×32) initial error covariance matrix of the extended state vector is partitioned as $\mathbf{P}_{0|0} = \begin{bmatrix} \mathbf{I}_{24} & \mathbf{0}_{24\times8} \\ \mathbf{0}_{8\times24} & 10^8\mathbf{I}_8 \end{bmatrix}$. The covariance matrices of both the measurement noise vector $\mathbf{v}$ and the system noise vector $\mathbf{w}$ are chosen to be $\mathbf{R} = 0.1\mathbf{I}_{10}$ and $\mathbf{Q} = 10^{-7}\mathbf{I}_{32}$, respectively.

Table 2
Comparison of the mean normalized 2-norm estimation errors for a 3-storey linear shear-beam building with traditional EKF and GEKF-UI.

Noise level	Method	c_1 (%)	k_1 (%)	c_2 (%)	k_2 (%)	c_3 (%)	k_3 (%)	f_e^* (%)
5%	Traditional EKF	184.13	92.25	169.81	88.21	207.74	90.57	N/A
	GEKF-UI	1.55	1.13	1.85	0.88	2.11	0.99	1.87
10%	Traditional EKF	281.36	151.92	228.33	125.56	332.77	141.41	N/A
	GEKF-UI	2.65	2.17	4.84	1.51	3.92	2.23	3.26

Note: N/A = not applicable as the tradition EKF cannot estimate the unknown inputs.

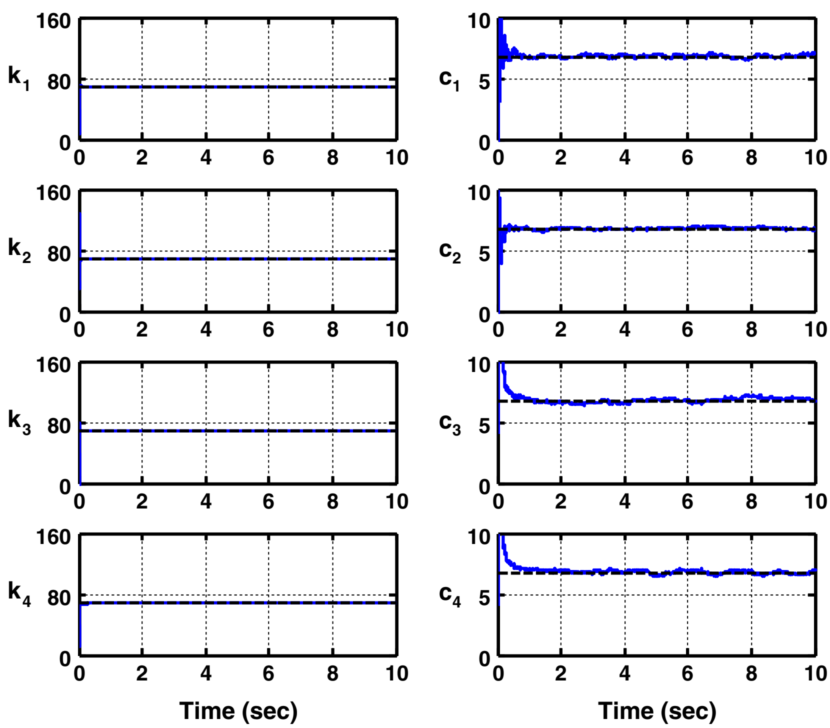

Fig. 4. Estimated parameters for the ASCE benchmark building. (Unit of k_i in MN/m and c_i in kN-s/m (i = 1, 2, 3, 4).) (—— estimated; ------ theoretical).

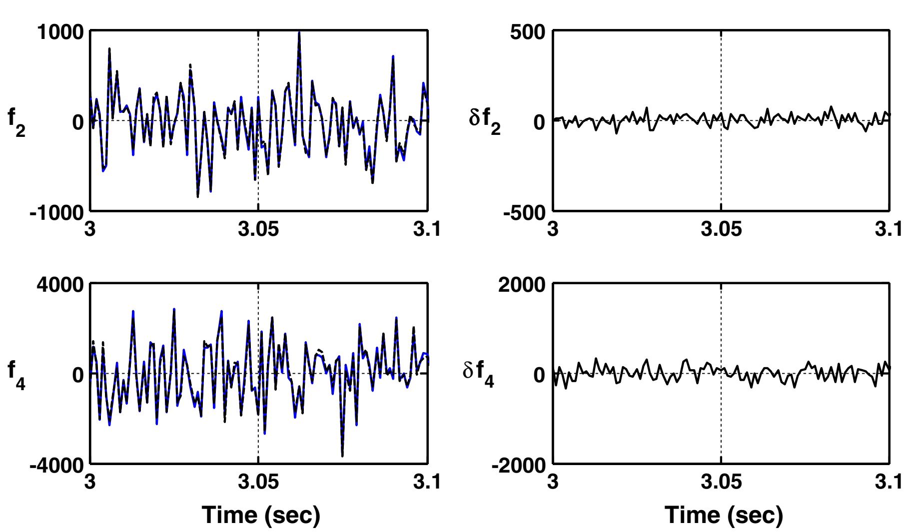

Fig. 5. Estimated unknown inputs for the ASCE benchmark building. (Unit of $f_2, f_4, \delta f_2$ and δf_4 in N.) (—— estimated; ------ theoretical).

Table 3
The mean normalized 2-norm estimation errors for ASCE benchmark building with GEKF-UI.

Noise level	c_1 (%)	k_1 (%)	c_2 (%)	k_2 (%)	c_3 (%)	k_3 (%)	c_4 (%)	k_4 (%)	f_2 (%)	f_4 (%)
5%	1.46	1.27	1.34	1.11	1.69	1.24	1.58	1.77	2.59	3.77
10%	3.12	2.93	3.02	2.45	3.37	2.77	3.31	3.65	5.48	7.12

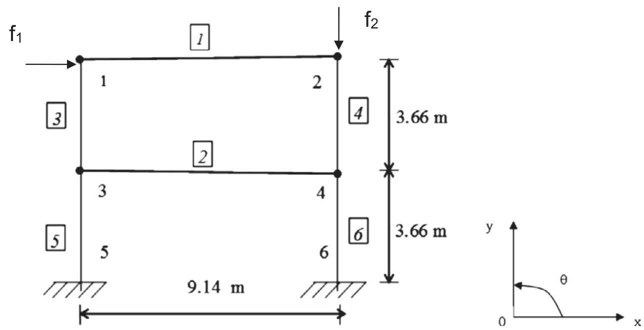

Fig. 6. Two-storey finite element model of planar frame structure.

Based on the tracking technique with the tolerance value $\delta = 10^{-2}$ for the optimization procedure to obtain the adaptive matrix $\mathbf{\Lambda}_{k+1}$ in Eq. (34) of Yang et al. [5]. The estimated parameters are presented in Fig. 7. The identified unknown excitations are presented in Fig. 8. Results from both figures indicate that the proposed technique is capable of tracking satisfactorily the system parameters and their changes as well as the unknown excitation.

4.4. Discussions on the simulation studies

(1) Results in Figs. 1–8 and Tables 2 and 3 demonstrate that the proposed GEKF-UI approach is capable of identifying the structural parameters and unknown inputs simultaneously from relative displacements and absolute acceleration measurements.

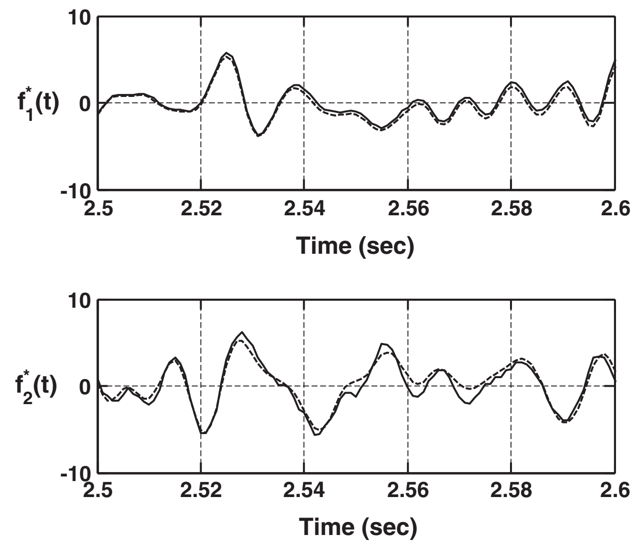

Fig. 8. Identified Unknown Inputs for the two-storey planar frame structure. (Unit of $f_1^*(t)$ and $f_2^*(t)$ in kN.) (—— estimated; ------ theoretical).

(2) Fig. 3 and Table 2 also show that the estimation using traditional EKF has large errors with noisy measurements.

(3) In the example with the 3-storey shear building, the application of EKF-UI [15] requires computation for the relative accelerations from the noisy inter-storey drifts by numerical differentiations and digital noise filters, which have to be executed off-line with additional errors. In this paper, the measured inter-storey drifts can be directly used on-line with the proposed GEKF-UI avoiding errors introduced in the numerical procedures.

(4) In the example with the ASCE benchmark building, the DOFs with excitations have to be measured according to Yang et al. [15], whereas only one DOF with excitation is measured with the proposed GEKF-UI. The GEKF-UI works well as noted in Figs. 4,5.

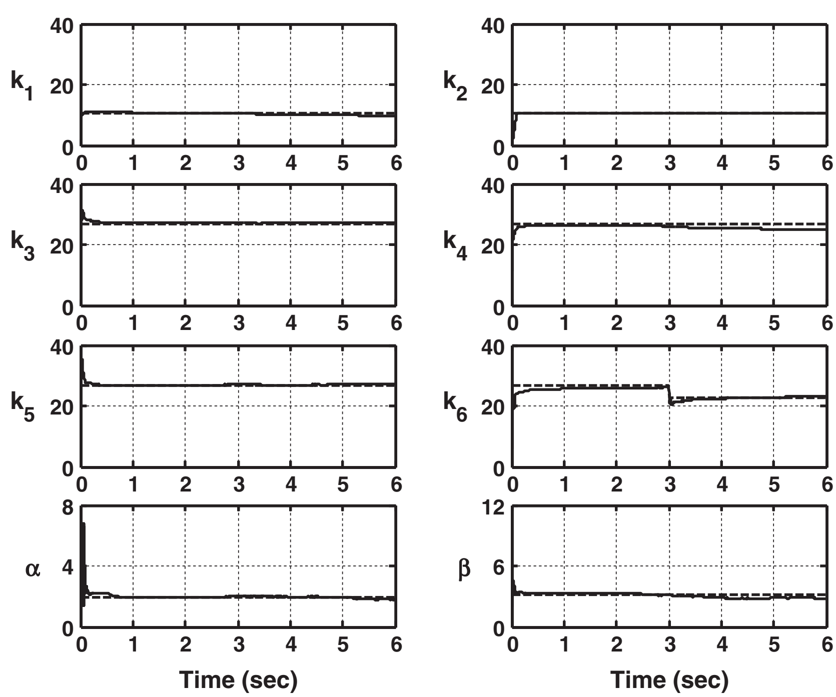

Fig. 7. Identified Parameters k_i ($i = 1, 2, 3, 4, 5, 6$), α and β for two-storey planar frame structure. (Unit of k_i in MN/m, α in m^2/s, and β in 10^{-4} s.) (—— estimated; ------ theoretical).

(5) The example with the two-storey planar frame structure demonstrates that the proposed GEKF-UI not only can deal with time-invariant parameter estimation, but it can also track the abrupt parameter variation by adopting an adaptive technique [5].
(6) Future research will include the application of the proposed GEKF-UI in structural health monitoring (SHM) systems, especially those for complex structures formulated with finite-element method.

5. Conclusions

A General Extended Kalman Filter with Unknown Inputs (GEKF-UI) approach is proposed to estimate the states and the unknown inputs simultaneously for the general case when the DOFs with unknown inputs are partially measured. The analytical recursive solution of GEKF-UI is derived by minimizing an objective function. The approach can be simplified into existing EKF approaches, such as the traditional EKF (e.g., [1]), the EKF-UI [15], and the EKF-UI-WDF [18], with different input scenarios. The proposed GEKF-UI is theoretically more flexible for different combinations of number, type and location of the sensors for the estimation of the structural parameters (as partial states) and the unknown inputs, compared with existing approaches [13–17]. Simulation results of a 3-DOFs linear damped shear building, the ASCE benchmark building, and a two-storey planar frame structure demonstrate the effectiveness of the proposed approach in identifying time-invariant and time-varying structural parameters as well as unknown inputs with different measurement scenarios with or without measurements at the excitation DOFs.

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Appendix A. Derivation of the recursive solution for $\hat{\mathbf{Z}}_{e,k+1|k+1}$

Following the pattern of formulation in [15], the states at ith time instant, $\mathbf{Z}_i$ ($i = 1, 2, \ldots, k$) can be expressed in terms of $\mathbf{Z}_{k+1}$ at (i+1) th time instant through repeated applications of the inverse transition relation in Eq. (11) as

$$\mathbf{Z}_i = \mathbf{\Phi}^{-1}_{k+1,i}\mathbf{Z}_{k+1} - \left\{\sum_{j=i}^{k}\mathbf{\Phi}^{-1}_{j+1,i}[\mathbf{B}^*_j\mathbf{f}^*_j + \mathbf{u}_{j|j}]\right\}; \quad i = 1, 2, \ldots, k \tag{A1}$$

where $\mathbf{\Phi}_{q,i} = \Pi_{j=i}^{q-1}\mathbf{\Phi}_{j+1,j}$ ($q = i + 1, \ldots, k + 1$) and $\mathbf{u}_{j|j}$ are given by Eq. (13) by replacing k with j.

Substituting the linearized $\mathbf{h}(\mathbf{Z}_i, \mathbf{f}_i, \mathbf{f}^*_i, i)$, which is given by Eq. (6) by replacing $k + 1$ with i for $i = 1, 2, \ldots, k + 1$), and $\mathbf{Z}_i$ from Eq. (A1) (for $i = 1, 2, \ldots, k$) into the first equation of Eq. (19), we have

$$\bar{\Delta}_{k+1} = \mathbf{Y}_{k+1} - \mathbf{A}_{e,k+1}\mathbf{Z}_{e,k+1} \tag{A2}$$

where $\mathbf{Z}_{e,k+1}$ is a $[(2m+n) + \bar{r}(k+1) + \tilde{r}k]$-extended unknown vector at $t = (k+1)\Delta t, \mathbf{Y}_{k+1}$ is a $l(k+1)$-known vector, and $\mathbf{A}_{e,k+1}$ is a $[l(k+1) \times ((2m+n) + \bar{r}(k+1) + \tilde{r}k)]$ known matrix, given by

$$\mathbf{Z}_{e,k+1} = \begin{bmatrix} \mathbf{Z}_{k+1} \\ \bar{\mathbf{f}}^*_1 \\ \tilde{\mathbf{f}}^*_1 \\ \bar{\mathbf{f}}^*_2 \\ \vdots \\ \tilde{\mathbf{f}}^*_k \\ \bar{\mathbf{f}}^*_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{k+1} \\ \bar{\mathbf{f}}^*_1 \\ \mathbf{f}^*_{e,2} \\ \vdots \\ \mathbf{f}^*_{e,k+1} \end{bmatrix}; \quad \mathbf{Y}_{k+1} = \begin{bmatrix} \bar{\mathbf{y}}_{1|k+1} \\ \bar{\mathbf{y}}_{2|k+1} \\ \vdots \\ \bar{\mathbf{y}}_{k+1|k+1} \end{bmatrix}; \quad \mathbf{A}_{e,k+1} = \begin{bmatrix} \tilde{\mathbf{L}}_{k+1} & \tilde{\mathbf{M}}_{k+1} \\ \tilde{\mathbf{H}}_{k+1} & \mathbf{D}^*_{k+1|k} \end{bmatrix} \tag{A3}$$

$\bar{\mathbf{y}}_{i|k+1}$ ($i = 1, 2, \ldots, k + 1$) in Eq. (A3) is as follows where $\tilde{\mathbf{L}}_{k+1}, \tilde{\mathbf{M}}_{k+1}, \tilde{\mathbf{H}}_{k+1}$ are given by

$$\tilde{\mathbf{L}}_{k+1} = \begin{bmatrix} \mathbf{H}_{1|0}\mathbf{\Phi}^{-1}_{k+1,1} & \mathbf{D}^*_{1|0} - \mathbf{H}_{1|0}\mathbf{\Phi}^{-1}_{2,1}\bar{\mathbf{B}}^*_1 & -\mathbf{H}_{1|0}\mathbf{\Phi}^{-1}_{2,1}\tilde{\mathbf{B}}^*_1 & \cdots & -\mathbf{H}_{1|0}\mathbf{\Phi}^{-1}_{k,1}\tilde{\mathbf{B}}^*_{k-1} & -\mathbf{H}_{1|0}\mathbf{\Phi}^{-1}_{k+1,1}\bar{\mathbf{B}}^*_k \\ \mathbf{H}_{2|1}\mathbf{\Phi}^{-1}_{k+1,2} & \mathbf{0} & \mathbf{0} & \mathbf{D}^*_{2|1} - \mathbf{H}_{2|1}\mathbf{\Phi}^{-1}_{3,2}\bar{\mathbf{B}}^*_2 & \cdots & -\mathbf{H}_{2|1}\mathbf{\Phi}^{-1}_{k+1,2}\bar{\mathbf{B}}^*_k \\ \mathbf{H}_{3|2}\mathbf{\Phi}^{-1}_{k+1,3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -\mathbf{H}_{3|2}\mathbf{\Phi}^{-1}_{k+1,3}\bar{\mathbf{B}}^*_k \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{H}_{k|k-1}\mathbf{\Phi}^{-1}_{k+1,k} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{D}^*_{k|k-1} - \mathbf{H}_{k|k-1}\mathbf{\Phi}^{-1}_{k+1,k}\bar{\mathbf{B}}^*_k \end{bmatrix}$$

$$\tilde{\mathbf{M}}_{k+1} = \begin{bmatrix} \mathbf{H}_{1|0}\mathbf{\Phi}^{-1}_{k+1,1}\tilde{\mathbf{B}}^*_k & \mathbf{0} \\ \mathbf{H}_{2|1}\mathbf{\Phi}^{-1}_{k+1,2}\tilde{\mathbf{B}}^*_k & \mathbf{0} \\ \mathbf{H}_{3|2}\mathbf{\Phi}^{-1}_{k+1,3}\tilde{\mathbf{B}}^*_k & \mathbf{0} \\ \vdots & \vdots \\ \mathbf{H}_{k|k-1}\mathbf{\Phi}^{-1}_{k+1,k}\tilde{\mathbf{B}}^*_k & \mathbf{0} \end{bmatrix}; \quad \tilde{\mathbf{H}}_{k+1} = \begin{bmatrix} \mathbf{H}_{k+1|k} & \mathbf{0} \end{bmatrix} \tag{A4}$$

where $\mathbf{B}^*_i = [\tilde{\mathbf{B}}^*_i \; \bar{\mathbf{B}}^*_i]$ corresponding to $\mathbf{f}^*_{i|i} = \begin{bmatrix}\tilde{\mathbf{f}}^{*\mathrm{T}}_{i|i} & \bar{\mathbf{f}}^{*\mathrm{T}}_{i|i}\end{bmatrix}^{\mathrm{T}}$ ($i = 1, 2, \ldots, k + 1$).

The following shows the derivation of the analytical recursive solutions for $\hat{\mathbf{Z}}_{k+1|k+1}$ and $\hat{\mathbf{f}}^*_{e,k+1|k+1}$, from the recursive solution for $\hat{\mathbf{Z}}_{e,k+1|k+1}$ (i.e., the relationship between $\hat{\mathbf{Z}}_{e,k+1|k+1}$ at $t = (k+1)\Delta t$ and $\hat{\mathbf{Z}}_{e,k|k}$ at $t = k\Delta t$). The estimation $\hat{\mathbf{Z}}_{e,k|k}$ for $\mathbf{Z}_{e,k}$ at $t = k\Delta t$ can be obtained from $\hat{\mathbf{Z}}_{e,k+1|k+1}$ in Eq. (20a) where $k + 1$ is replaced by k, as

$$\hat{\mathbf{Z}}_{e,k|k} = \mathbf{P}_{e,k}[\mathbf{A}^{\mathrm{T}}_{e,k}\mathbf{W}_k\mathbf{Y}_k]; \quad \mathbf{P}_{e,k} = [\mathbf{A}^{\mathrm{T}}_{e,k}\mathbf{W}_k\mathbf{A}_{e,k}]^{-1} \tag{A5}$$

where

$$\mathbf{Z}_{e,k} = \begin{bmatrix} \mathbf{Z}_k \\ \bar{\mathbf{f}}_1^* \\ \tilde{\mathbf{f}}_1^* \\ \bar{\mathbf{f}}_2^* \\ \vdots \\ \tilde{\mathbf{f}}_{k-1}^* \\ \bar{\mathbf{f}}_k^* \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_k \\ \bar{\mathbf{f}}_1^* \\ \mathbf{f}_{e,2}^* \\ \vdots \\ \mathbf{f}_{e,k}^* \end{bmatrix}; \qquad \mathbf{Y}_k = \begin{bmatrix} \bar{\mathbf{y}}_{1|k} \\ \bar{\mathbf{y}}_{2|k} \\ \vdots \\ \bar{\mathbf{y}}_{k|k} \end{bmatrix}; \qquad \mathbf{A}_{e,k} = \begin{bmatrix} \tilde{\mathbf{L}}_k & \tilde{\mathbf{M}}_k \\ \tilde{\mathbf{H}}_k & \mathbf{D}_{k|k-1}^* \end{bmatrix} \tag{A6}$$

where $\mathbf{W}_k = diag(\mathbf{R}_1^{-1}, \mathbf{R}_2^{-1}, \ldots, \mathbf{R}_k^{-1})$ with $\mathbf{R}_i = E[\mathbf{v}_i \mathbf{v}_i^{\mathrm{T}}]$ ($i = 1, 2, \ldots, k$).

With the analysis of the matrix structure, the relationship between $\mathbf{A}_{e,k+1}$ and $\mathbf{A}_{e,k}$ can be found for $k = 1, 2, \ldots$ as

$$\mathbf{A}_{e,k+1} = \begin{bmatrix} \mathbf{A}_{e,k}\mathbf{\Phi}_{k+1,k}^{*-1} & -\mathbf{A}_{e,k}\mathbf{\Phi}_{k+1,k}^{*-1}\bar{\mathbf{M}}_k \\ \tilde{\mathbf{H}}_{k+1} & \mathbf{D}_{k+1|k}^* \end{bmatrix}$$

$$\mathbf{\Phi}_{k+1,k}^{*-1} = \begin{bmatrix} \mathbf{\Phi}_{k+1,k}^{-1} & -\mathbf{\Phi}_{k+1,k}^{-1}\bar{\mathbf{B}}_k \\ \mathbf{0} & \mathbf{I} \end{bmatrix}; \qquad \bar{\mathbf{M}}_k = \begin{bmatrix} \tilde{\mathbf{B}}_k^* & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}; \qquad \bar{\mathbf{B}}_k = [\mathbf{0} \;\vdots\; \bar{\mathbf{B}}_k^*] \tag{A7}$$

The relationship between $\mathbf{Y}_{k+1}$ and $\mathbf{Y}_k$, as well as $\mathbf{W}_{k+1}$ and $\mathbf{W}_k$ can be found in a similar manner as

$$\mathbf{Y}_{k+1} = \begin{bmatrix} \mathbf{Y}_k + \mathbf{A}_{e,k}\mathbf{\Phi}_{k+1,k}^{*-1}\tilde{\mathbf{F}}_k \\ \bar{\mathbf{y}}_{k+1|k+1} \end{bmatrix}; \qquad \tilde{\mathbf{F}}_k = \begin{bmatrix} \mathbf{u}_{k|k} \\ \mathbf{0} \end{bmatrix}; \qquad \mathbf{W}_{k+1} = \begin{bmatrix} \mathbf{W}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{k+1}^{-1} \end{bmatrix} \tag{A8}$$

It follows from Eq. (20) that, $\mathbf{P}_{e,k+1}$ is the inversion of matrix $\mathbf{A}_{e,k+1}^{\mathrm{T}}\mathbf{W}_{k+1}\mathbf{A}_{e,k+1}$. With the aid of two matrix decomposition formulas [15], $\mathbf{P}_{e,k+1}$ in Eq. (20) can be transformed as follows in a similar way,

$$\mathbf{P}_{e,k+1} = \begin{bmatrix} \begin{matrix} \bar{\mathbf{P}}_{e,k+1} + \bar{\mathbf{P}}_{e,k+1}(\tilde{\mathbf{H}}_{k+1}^{\mathrm{T}}\mathbf{R}_{k+1}^{-1}\bar{\mathbf{D}}_{k+1|k}^{*}\mathbf{S}_{k+1}\bar{\mathbf{D}}_{k+1|k}^{*\mathrm{T}}\mathbf{R}_{k+1}^{-1}\tilde{\mathbf{H}}_{k+1} \\ +\tilde{\mathbf{P}}_{e,k+1}^{-1}\bar{\mathbf{B}}_{k+1}\mathbf{S}_{k+1}\bar{\mathbf{B}}_{k+1}^{\mathrm{T}}\tilde{\mathbf{P}}_{e,k+1}^{-1} + \tilde{\mathbf{H}}_{k+1}^{\mathrm{T}}\mathbf{R}_{k+1}^{-1}\bar{\mathbf{D}}_{k+1|k}^{*}\mathbf{S}_{k+1}\bar{\mathbf{B}}_{k+1}^{\mathrm{T}}\tilde{\mathbf{P}}_{e,k+1}^{-1} \\ +\tilde{\mathbf{P}}_{e,k+1}^{-1}\bar{\mathbf{B}}_{k+1}\mathbf{S}_{k+1}\bar{\mathbf{D}}_{k+1|k}^{*\mathrm{T}}\mathbf{R}_{k+1}^{-1}\tilde{\mathbf{H}}_{k+1})\bar{\mathbf{P}}_{e,k+1} \end{matrix} & \bar{\mathbf{P}}_{e,k+1}(\tilde{\mathbf{P}}_{e,k+1}^{-1}\bar{\mathbf{M}}_k - \tilde{\mathbf{H}}_{k+1}^{\mathrm{T}}\mathbf{R}_{k+1}^{-1}\mathbf{D}_{k+1|k}^{*})\mathbf{S}_{k+1} \\ \mathbf{S}_{k+1}(\tilde{\mathbf{P}}_{e,k+1}^{-1}\bar{\mathbf{M}}_k - \tilde{\mathbf{H}}_{k+1}^{\mathrm{T}}\mathbf{R}_{k+1}^{-1}\mathbf{D}_{k+1|k}^{*})^{\mathrm{T}}\bar{\mathbf{P}}_{e,k+1} & \mathbf{S}_{k+1} \end{bmatrix} \tag{A9}$$

where

$$\bar{\mathbf{P}}_{e,k+1} = [\mathbf{I} - \mathbf{K}_{e,k+1}\tilde{\mathbf{H}}_{k+1}]\tilde{\mathbf{P}}_{e,k+1} \tag{A10}$$

$$\mathbf{S}_{k+1} = \{(\mathbf{D}_{k+1|k}^* - \tilde{\mathbf{H}}_{k+1}\bar{\mathbf{M}}_k)^{\mathrm{T}}[\mathbf{R}_{k+1} + \tilde{\mathbf{H}}_{k+1}\tilde{\mathbf{P}}_{e,k+1}\tilde{\mathbf{H}}_{k+1}^{\mathrm{T}}]^{-1}(\mathbf{D}_{k+1|k}^* - \tilde{\mathbf{H}}_{k+1}\bar{\mathbf{M}}_k)\}^{-1} \tag{A11}$$

where

$$\tilde{\mathbf{P}}_{e,k+1} = \mathbf{\Phi}_{k+1,k}^*\mathbf{P}_{e,k}\mathbf{\Phi}_{k+1,k}^{*\mathrm{T}} \tag{A12}$$

$$\mathbf{K}_{e,k+1} = \tilde{\mathbf{P}}_{e,k+1}\tilde{\mathbf{H}}_{k+1}^{\mathrm{T}}[\mathbf{R}_{k+1} + \tilde{\mathbf{H}}_{k+1}\tilde{\mathbf{P}}_{e,k+1}\tilde{\mathbf{H}}_{k+1}^{\mathrm{T}}]^{-1} \tag{A13}$$

$$\mathbf{\Phi}_{k+1,k}^* = \begin{bmatrix} \mathbf{\Phi}_{k+1,k} & \bar{\mathbf{B}}_k \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{A14}$$

where $\bar{\mathbf{B}}_k$ is given by Eq. (A7) and $\mathbf{P}_{e,k}$ is given by Eq. (A5).

Substituting Eqs. (A9)–(A14) into the first equation of Eq. (20), one obtains the recursive solution for $\hat{\mathbf{Z}}_{e,k+1|k+1}$ as

$$\hat{\mathbf{Z}}_{e,k+1|k+1} = \begin{bmatrix} \bar{\mathbf{Z}}_{e,k+1} + \bar{\mathbf{P}}_{e,k+1}(\tilde{\mathbf{P}}_{e,k+1}^{-1}\bar{\mathbf{B}}_k - \tilde{\mathbf{H}}_{k+1}^{\mathrm{T}}\mathbf{R}_{k+1}^{-1}\mathbf{D}_{k+1|k}^*)\hat{\mathbf{f}}_{e,k+1|k+1}^* \\ \hat{\mathbf{f}}_{e,k+1|k+1}^* \end{bmatrix} \tag{A15}$$

where

$$\bar{\mathbf{Z}}_{e,k+1} = (\mathbf{\Phi}_{k+1,k}^*\hat{\mathbf{Z}}_{e,k|k} + \tilde{\mathbf{F}}_k) + \mathbf{K}_{e,k+1}[\bar{\mathbf{y}}_{k+1|k+1} - \tilde{\mathbf{H}}_{k+1}(\mathbf{\Phi}_{k+1,k}^*\hat{\mathbf{Z}}_{e,k|k} + \tilde{\mathbf{F}}_k)] \tag{A16}$$

$$\hat{\mathbf{f}}_{e,k+1|k+1}^* = \mathbf{S}_{k+1}\bar{\mathbf{D}}_{k+1|k}^{*\mathrm{T}}\mathbf{R}_{k+1}^{-1}(\bar{\mathbf{y}}_{k+1|k+1} - \tilde{\mathbf{H}}_{k+1}\bar{\mathbf{Z}}_{e,k+1}) + \mathbf{S}_{k+1}\bar{\mathbf{M}}_k^{\mathrm{T}}\tilde{\mathbf{P}}_{e,k+1}^{-1}\mathbf{K}_{e,k}[\bar{\mathbf{y}}_{k+1|k+1} - \tilde{\mathbf{H}}_{k+1}(\mathbf{\Phi}_{k+1,k}^*\hat{\mathbf{Z}}_{e,k|k} + \tilde{\mathbf{F}}_k)] \tag{A17}$$

Appendix B. Derivation of the recursive solution for $\hat{\mathbf{Z}}_{k+1|k+1}$ and $\hat{\mathbf{f}}_{e,k+1|k+1}^*$

The recursive solution for the estimate $\hat{\mathbf{Z}}_{e,k+1|k+1}$ has been derived in Appendix A. It should be noted that the dimension of $\hat{\mathbf{Z}}_{e,k+1|k+1}$ increases rapidly as the time instant k increases with increasing computational effort. In this Appendix, the recursive solutions for $\hat{\mathbf{Z}}_{k+1|k+1}$ and $\hat{\mathbf{f}}_{e,k+1|k+1}^*$ are derived from the recursive solution for $\hat{\mathbf{Z}}_{e,k+1|k+1}$ obtained in Appendix A with further decompositions.

From the definition of $\hat{\mathbf{Z}}_{e,k+1|k+1}$ given in Eq. (20b), $\hat{\mathbf{Z}}_{e,k+1|k+1}$ can be partitioned as follows.

$$\hat{\mathbf{Z}}_{e,k+1|k+1} = \left[\hat{\mathbf{Z}}_{k+1|k+1}^{\mathrm{T}} \;\vdots\; \hat{\mathbf{F}}_{k|k+1}^{*\mathrm{T}} \;\vdots\; \hat{\mathbf{f}}_{e,k+1|k+1}^{*\mathrm{T}}\right]^{\mathrm{T}} \tag{B1}$$

where $\hat{\mathbf{F}}_{k|k+1}^* = \left[\hat{\bar{\mathbf{f}}}_{1|k+1}^{*\mathrm{T}} \;\vdots\; \hat{\tilde{\mathbf{f}}}_{1|k+1}^{*\mathrm{T}} \;\vdots\; \hat{\bar{\mathbf{f}}}_{2|k+1}^{*\mathrm{T}} \;\vdots\; \cdots \;\vdots\; \hat{\bar{\mathbf{f}}}_{k|k+1}^{*\mathrm{T}}\right]^{\mathrm{T}}$. Comparing Eq. (B1) and Eq. (A15), one obtains

$$\begin{bmatrix} \hat{\mathbf{Z}}_{k+1|k+1} \\ \hat{\mathbf{F}}_{k|k+1}^* \end{bmatrix} = \bar{\mathbf{Z}}_{e,k+1} + \bar{\mathbf{P}}_{e,k+1}(\tilde{\mathbf{P}}_{e,k+1}^{-1}\bar{\mathbf{B}}_k - \tilde{\mathbf{H}}_{k+1}^{\mathrm{T}}\mathbf{R}_{k+1}^{-1}\mathbf{D}_{k+1|k}^*)\hat{\mathbf{f}}_{e,k+1|k+1}^* \tag{B2}$$

where $\hat{\mathbf{f}}_{e,k+1|k+1}^*$ is given by Eq. (A17).

Thus, the recursive solution for $\hat{\mathbf{Z}}_{k+1|k+1}$ can be obtained from the top $(2m+n)$ elements of the right side of Eq. (B2). Further, the recursive solution for $\hat{\mathbf{f}}_{e,k+1|k+1}^*$ has been obtained from Eq. (A17), which can be further simplified. To decompose the right side of Eq. (B2) and to simplify $\hat{\mathbf{f}}_{e,k+1|k+1}^*$ in Eq. (A17), $\hat{\mathbf{Z}}_{e,k|k}$ and $\mathbf{P}_{e,k}$ in Eq. (A5) are partitioned as

$$\hat{\mathbf{Z}}_{e,k|k} = \begin{bmatrix} \hat{\mathbf{Z}}_{k|k} \\ \hat{\mathbf{F}}^*_{k|k} \end{bmatrix}; \qquad \mathbf{P}_{e,k} = \begin{bmatrix} \mathbf{P}_{\mathbf{Z},k|k} & \mathbf{P}_{\mathbf{Z}\mathbf{f}^*_e,k|k} \\ \mathbf{P}_{\mathbf{f}^*_e\mathbf{Z},k|k} & \mathbf{P}_{\mathbf{f}^*_e,k|k} \end{bmatrix} \tag{B3}$$

where $\hat{\mathbf{F}}^*_{k|k} = [\hat{\hat{\mathbf{f}}}^{*\mathrm{T}}_{1|k} \; \hat{\mathbf{f}}^{*\mathrm{T}}_{1|k} \; \hat{\mathbf{f}}^{*\mathrm{T}}_{2|k} \; \cdots \; \hat{\mathbf{f}}^{*\mathrm{T}}_{k-1|k} \; \hat{\mathbf{f}}^{*\mathrm{T}}_{k|k}]^{\mathrm{T}}$ is rk-vector and $\mathbf{P}_{\mathbf{Z},k|k}$ is $[(2m+n)\times(2m+n)]$ matrix.

With the aid of Eq. (B3), the decompositions for $\tilde{\mathbf{P}}_{e,k+1}, \mathbf{K}_{e,k+1}, \bar{\mathbf{Z}}_{e,k+1}$ and $\bar{\mathbf{P}}_{e,k+1}$ [15] can be executed and the recursive solution for $\hat{\mathbf{Z}}_{k+1|k+1}$ and $\hat{\mathbf{f}}^*_{e,k+1|k+1} = \begin{bmatrix} \hat{\hat{\mathbf{f}}}^*_{k|k+1} \\ \hat{\hat{\mathbf{f}}}^*_{k+1|k+1} \end{bmatrix}$ can be obtained by substituting the decomposed quantities into Eqs. (B2) and (A17), respectively.

Also, $\mathbf{K}_{\mathbf{Z},k+1}$ in Eq. (25) can be obtained by changing only the notation $\tilde{\mathbf{P}}_{e11,k+1}$ to the notation $\mathbf{P}_{\mathbf{Z},k+1|k}$ in the top set of partitioned equations of Eq. (B3), i.e., introducing the new quantity $\mathbf{P}_{\mathbf{Z},k+1|k}$ as

$$\mathbf{P}_{\mathbf{Z},k+1|k} = \tilde{\mathbf{P}}_{e11,k+1} = \mathbf{\Phi}_{k+1,k}\mathbf{P}_{\mathbf{Z},k|k}\mathbf{\Phi}^{\mathrm{T}}_{k+1,k} + \mathbf{Q}_{k+1} \tag{B4}$$

where $\mathbf{P}_{\mathbf{Z},k+1|k}$ is a $[(2m+n)\times(2m+n)]$ matrix.

The recursive solution $\mathbf{P}_{\mathbf{Z},k+1|k}$ presented in Eq. (B4) involves $\mathbf{P}_{\mathbf{Z},k|k}$ which is a sub-matrix of $\mathbf{P}_{e,k}$ in Eq. (A5). The recursive solution for $\mathbf{P}_{\mathbf{Z},k|k}$ can be obtained from the recursive solution of $\mathbf{P}_{\mathbf{Z},k+1|k+1}$ by replacing $k+1$ by k.

Similar to Eq. (B3), $\mathbf{P}_{e,k+1}$ in Eq. (20) is further partitioned as follows

$$\mathbf{P}_{e,k+1} = \begin{bmatrix} \mathbf{P}_{\mathbf{Z},k+1|k+1} & \mathbf{P}_{\mathbf{Z}\mathbf{f}^*_e,k+1|k+1} \\ \mathbf{P}_{\mathbf{f}^*_e\mathbf{Z},k+1|k+1} & \mathbf{P}_{\mathbf{f}^*_e,k+1|k+1} \end{bmatrix} \tag{B5}$$

where $\mathbf{P}_{\mathbf{Z},k+1|k+1}$ is a $[(2m+n)\times(2m+n)]$ matrix corresponding to $\mathbf{Z}_{k+1}$. It is observed from Eqs. (B5) and (A9) that $\mathbf{P}_{e,k+1}$ has been partitioned into different sub-matrices with different dimensions. Let the upper-left sub-matrix of Eq. (A9) be denoted by $\mathbf{P}_{e11,k+1}$, which is a $[((2m+n)+rk)\times((2m+n)+rk)]$ matrix, i.e.,

$$\begin{aligned}\mathbf{P}_{e11,k+1} = {} & \bar{\mathbf{P}}_{e,k+1} + \bar{\mathbf{P}}_{e,k+1}(\tilde{\mathbf{H}}^{\mathrm{T}}_{k+1}\mathbf{R}^{-1}_{k+1}\bar{\mathbf{D}}^*_{k+1|k}\mathbf{S}_{k+1}\bar{\mathbf{D}}^{*\mathrm{T}}_{k+1|k}\mathbf{R}^{-1}_{k+1}\tilde{\mathbf{H}}_{k+1} \\ & + \tilde{\mathbf{P}}^{-1}_{e,k+1}\bar{\mathbf{B}}_{k+1}\mathbf{S}_{k+1}\bar{\mathbf{B}}^{\mathrm{T}}_{k+1}\tilde{\mathbf{P}}^{-1}_{e,k+1} \\ & + \tilde{\mathbf{H}}^{\mathrm{T}}_{k+1}\mathbf{R}^{-1}_{k+1}\bar{\mathbf{D}}^*_{k+1|k}\mathbf{S}_{k+1}\bar{\mathbf{B}}^{\mathrm{T}}_{k+1}\tilde{\mathbf{P}}^{-1}_{e,k+1} \\ & + \tilde{\mathbf{P}}^{-1}_{e,k+1}\bar{\mathbf{B}}_{k+1}\mathbf{S}_{k+1}\bar{\mathbf{D}}^{*\mathrm{T}}_{k+1|k}\mathbf{R}^{-1}_{k+1}\tilde{\mathbf{H}}_{k+1})\bar{\mathbf{P}}_{e,k+1}\end{aligned} \tag{B6}$$

A comparison of the dimension of $\mathbf{P}_{\mathbf{Z},k+1|k+1}$ in Eq. (B5) with that of $\mathbf{P}_{e11,k+1}$ in Eq. (B6) indicates that $\mathbf{P}_{\mathbf{Z},k+1|k+1}$ is the upper-left sub-matrix of $\mathbf{P}_{e11,k+1}$. Then, the recursive solution for $\mathbf{P}_{\mathbf{Z},k+1|k+1}$ is obtained by substituting the decomposed $\bar{\mathbf{P}}_{e,k+1}$ into Eq. (B6) and extracting the $[(2m+n)\times(2m+n)]$ upper-left sub-matrix, $\mathbf{P}_{\mathbf{Z},k+1|k+1}$, from the resulting $\mathbf{P}_{e11,k+1}$.

Finally, $\mathbf{S}_{k+1}$ in Eq. (A11) can be simplified by using the decomposed $\bar{\mathbf{P}}_{e,k+1}$ and $\tilde{\mathbf{H}}_{k+1}$. As a summary, the derivation for the analytical recursive solution for GEKF-UI has been completed as shown in Eqs. (21)–(28).

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