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Review

Boundedly rational route choice behavior: A review of models and methodologies

Xuan Di, Henry X. Liu*

Department of Civil and Environmental Engineering, University of Michigan Transportation Research Institute, University of Michigan, Ann Arbor, United States

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ABSTRACT

Perfect rationality (PR) has been widely used in modeling travel behavior. As opposed to PR, bounded rationality (BR) has recently regained researchers' attention since it was first introduced into transportation science in the 1980s due to its power in more realistic travel behavior modeling and prediction. This paper provides a comprehensive survey on the models of BR route choice behavior, aiming to identify current research gaps and provide directions for future research. Despite a small but growing body of studies on employing bounded rationality principle, BR route choice behavior remains understudied due to the following reasons: (a) The existence of BR thresholds leads to mathematically intractable properties of equilibria; (b) BR parameters are usually latent and difficult to identify and estimate; and (c) BR is associated with human being's cognitive process and is challenging to model. Accordingly, we will review how existing literature addresses the aforementioned challenges in substantive and procedural bounded rationality models. Substantive bounded rationality models focus on choice outcomes while procedural bounded rationality models focus on the empirical studies of choice processes. Bounded rationality models in each category can be further divided based on whether time dimension is included. Accordingly, static and dynamic traffic assignment are introduced in substantive bounded rationality while two-stage cognitive models and day-to-day learning models in procedural bounded rationality are discussed. The methodologies employed in substantive bounded rationality include game theory and interactive congestion game, while those in procedural bounded rationality mainly adopt random utility and non- or semi-compensatory models. A comparison of all existing methodologies are given and bounded rationality models' scope and boundaries in terms of predictability, transferability, tractability, and scalability are discussed. Finally existing research gaps are presented and several promising future research directions are given.

1. Introduction

Perfect rationality is widely used in modeling travelers' decision-making behavior. For instance, in mode choice, travelers are assumed to be expected disutility minimizers (Ben-Akiva and Lerman, 1985); and in route choice, only the paths with the least disutility or the least generalized cost are chosen (Sheffi, 1984). As opposed to 'rationality as optimization',

* Corresponding author. Tel.: +1 734 764 4354.
E-mail address: henryliu@umich.edu (H.X. Liu).

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Simon (1957) proposed that people are boundedly rational in their decision-making processes and they tend to seek a satisfactory choice solution instead. This is either because people lack accurate information, or they are incapable of obtaining an optimized decision due to complexity of the situations. Bounded rationality also requires less computational burdens and ensures existence of a satisficing solution. People search decisions dynamically and will not terminate till an alternative meeting a certain threshold level is found. This level will be adjusted if a satisficing alternative is difficult to find. “Such changes in aspiration level...tend to guarantee the existence of satisfactory solutions” (Simon, 1957). Due to its prevalence in human behavior, ‘bounded rationality’ has been studied extensively in economics and psychology.

Introduction of bounded rationality into transportation science originated from the need to explain experimental findings of travel behavior which cannot be captured by perfectly rational modeling. Mahmassani and Chang (1987) first employed bounded rationality (BR) in modeling pre-trip departure time selection for a single bottleneck. Since then, there is “small but growing” (Ridwan, 2004) literature on incorporating bounded rationality into various transportation models, such as hyperpath assignment (Fonzone and Bell, 2010), dynamic traffic assignment (Szeto and Lo, 2006), transportation planning (Gifford and Checherita, 2007; Khisty and Arslan, 2005), traffic policy making (Marsden et al., 2012) and traffic safety (Sivak, 2002). All these studies indicate that the BR assumption plays an important role in transportation modeling. However, “there is not yet much convergence among them” (Ridwan, 2004). In other words, there does not exist a standard BR framework for travel behavior study.

In this paper, we aim to conduct a comprehensive survey on boundedly rational travel behavior. There are two types of behavioral research (Simon, 1982): “studies that are aimed at discovering and testing invariant laws of human individual or social behavior” and “studies that estimate parameters we need for fitting theoretical models incorporating known/believed laws to particular situations where we wish to make predictions”. The former is to reveal behavior and the latter is to model behavior. Accordingly, we will first review behavioral studies on disclosing and verifying bounded rationality. Then we will summarize research on boundedly rational route choice behavior models.

The rest of the paper is organized as follows: in Section 2, empirical and experimental evidence is listed to support bounded rationality in modeling people’s choice behavior. An overview of boundedly rational route choice models will be first summarized in Section 3. In Sections 4.1.1–7, BR formulations are introduced in static traffic assignment, dynamic traffic assignment, two-stage cognitive process and learning models. An in-depth discussion of the boundaries of bounded rationality models along with selection criteria are discussed in Section 8. The present research gaps are summarized and several promising future research directions are pinpointed in Section 9.

2. Behavioral evidence on bounded rationality

In this section, we will review existing empirical evidence to show that perfect rationality is too ideal and boundedly rational behavioral framework is needed.

2.1. Why not perfect rationality

2.1.1. Heuristic and bias

Psychologists and experimental economists verify that people use heuristic rules when making decisions, leading to biases or systematic errors (Conlisk, 1996). For example, people react differently under the same situations when the problem is presented in different ways, called “framing effect” (Tversky et al., 1981).

‘Debiasing’ experiments are conducted to test whether biases caused by heuristic processes can be eliminated through repeated practice and adequate incentives or punishments. However, several research indicates that biases are “substantial and important behavioral regularities” (Conlisk, 1996) and will not disappear due to deliberation costs.

On the other hand, heuristics are also critical tools people employ when making decisions. People try to tradeoff “between cognitive effort and judgemental accuracy”. Due to high costs of deliberation and information search, people tend to use heuristics to find the first alternative which they are satisfied with instead of calculating an optimal one.

2.1.2. Cognitive limit and deliberation cost

Hiraoka et al. (2002) showed that cognitive limits and deliberation costs play important roles in route choices. An experiment was designed where subjects spoke aloud while choosing routes and a protocol analysis was conducted to analyze subjects’ cognitive processes from verbal data. Results indicate that drivers have the desire to choose routes with less travel time, involving less cognitive resources and making them feel comfortable while driving along. Among the above three route choice criteria, a choice consuming less cognitive process dominates the other two criteria and drivers choose routes dynamically when one route satisfying their criteria is found.

2.1.3. Violation of taking shortest paths

Transportation researchers from across the world have found evidence that people do not usually take the shortest paths and the utilized paths often have higher costs than shortest ones.

After evaluating habitual routes, only 59% respondents from Cambridge, Massachusetts (Bekhor et al., 2006), 30% from Boston (Ramming, 2001), 87% from Turin, Italy (Prato and Bekhor, 2006) chose paths with the shortest distance or the shortest travel time. According to GPS studies, 60% of subject commuters in the Twin Cities, Minnesota took paths longer than the shortest travel time paths (Zhu, 2011) and high percentage of commuting routes in Nagoya, Japan (Morikawa et al., 2005) and Lexington, Kentucky (Jan et al., 2000) were found to differ considerably from the shortest paths.

2.1.4. Nonexistence of perfect rationality via learning processes

Some opponents in economics claim that people can improve their rationality via repeated learning process. In other words, people can approach unbounded rationality while making decisions everyday based on previous experiences. Conlisk (1996) argued that learning mechanism does improve people's decision-making towards the optimal in some situations, but it can also hinder learning and adaptation due to habit. This argument has been supported by a sequence of route choice experiments. For example, compared to unfamiliar drivers, familiar drivers stick mostly to their usual driving routes which may be longer than the shortest path (Lotan, 1997). On the other hand, drivers familiar to the destination may adapt en-route choices dynamically based on real-time traffic information (Hiraoka et al., 2002).

In summary, all above statements show that perfect rationality cannot capture people's cognitive processes in decision-making and more realistic assumption is needed in travel behavior modeling.

2.2. Why bounded rationality

Perfectly rational models cause estimation and prediction errors. It is thus imperative to have a new paradigm which can explain empirical findings deviating from perfect rationality. The theory of bounded rationality has the capability of capturing observed deviations by considering people's cognitive limits and deliberation costs, habits and myopia.

2.2.1. Habit and inertia

People "place higher value on an opportunity if it is associated with the status quo" (Samuelson and Zeckhauser, 1988), because it can provide significant energy saving to cognitive thinking. Much empirical evidence suggests that habit plays a significant role in people's behavior in stable situations (Bamberg and Schmidt, 2003).

Habit may result from searching for an optimal solution in prevailing circumstances, but it also prevents people from pursuing better alternatives when situation changes and can collapse to "bad habit" (Jager, 2003). Lotan (1997) compared the impact of information on familiar and unfamiliar drivers. Ten familiar drivers and fifteen unfamiliar drivers were selected to drive in the Newton network in Massachusetts coded in traffic simulators. Results indicated that familiar drivers were reluctant to receive new information and only considered salient information. Therefore most of them stuck to their usual driving routes and did not necessarily minimize travel time.

Habit can be represented by a threshold in modeling travel choices. Cantillo et al. (2007, 2006) applied a discrete choice model with thresholds to simulated SP/RP mode choice datasets and showed that a model not considering inertia overestimates the benefits of transport investments substantially. Lotan (1997) fit an approximate-reasoning based model and a random utility model respectively to driving simulation data aiming to estimate and predict route choices. Results showed that the approximate-reasoning based model outperforms the random utility model. Carrion and Levinson (2012) studied commuters' day-to-day route choices from GPS data collected from 65 subjects for about 30 days, concluding that commuters chose routes based on a specific threshold and might abandon a route if its travel time exceeded the margin.

Mahmassani and his colleagues conducted a series of route choice experiments in the 1990s showing that even when all path cost information was available to travelers, commuters would not switch to shorter paths due to existence of inertia, which was quantified by the 'indifference band' (Hu and Mahmassani, 1997; Jayakrishnan et al., 1994; Mahmassani and Chang, 1987; Mahmassani and Jayakrishnan, 1991; Mahmassani and Liu, 1999; Srinivasan and Mahmassani, 1999). Accordingly a boundedly rational route choice framework was proposed to capture people's travel behavior with information provision. By comparing commuter departure time and route choice switch behavior in laboratory experiments with field surveys in Dallas and Austin, Texas, Mahmassani and Jou (2000) showed that boundedly rational route choice modeling observed from experiments provided a valid description of actual commuter daily behavior.

2.2.2. Myopia

Myopia refers to the fact that people do not usually concern for wider interests or longer-term consequences while making decisions. Consumers manifest myopia when purchasing large appliances and tend to buy models with lower price but higher energy consumption (Conlisk, 1996).

Similarly when making travel choices, travelers tend to switch to a link at an intersection which seems shorter for the time being but may lead to a longer route. Recent travel experiences also impact people's travel choice more profoundly. Bogers et al. (2005) used an interactive travel simulator "TSL" developed by Delft University of Technology to investigate route choice behavior. Subjects were asked to make route choices among two alternative paths for 25 simulation days. En route information was provided by a built-in dynamic traffic model and realized travel times were given in three different scenarios: travel time on the chosen route for the latest period, travel times on both routes for the latest period, and travel times on both routes for all past periods. Experiential results showed that more weights were given to previous day's travel experiences, i.e. lateness in minutes (weights were -2.41 when only the previous day's travel time was provided v.s. -1.4 when all past periods' experienced travel time was provided). Therefore human being's limited memory partially leads to myopia.

3. Boundedly rational route choice

Bounded rationality (BR) is rather a more realistic behavioral foundation than a new theory. Thus it permeates every part of travel behavioral modeling. Due to its power in more realistic travel behavior modeling and prediction, bounded rationality (BR) has regained researchers' attention recently since it was first introduced into transportation in the 1980s.

Despite a small but growing body of studies on employing bounded rationality principle in transport research, BR route choice behavior still remains understudied because route choice, compared to other travel choices such as mode choice or departure-time choice, is more challenging:

- One traveler's feasible path choice set can be huge.
- Factors attributable to path choice can be numerous.
- Major attributes associated with one path, i.e., travel time, travel reliability, are stochastic in nature and thus not easy to quantify.
- Other attributes factoring in route choice, such as congestion level, scenery, and other psychological factors, are complex.
- Route choice involves spatial dimensions of networks, which makes the choice process more complex.
- Many paths overlap with each other for a majority of portions. It is not always easy to distinguish them like mode choices. In other words, path choice cannot be directly treated as discrete choice.

In addition to the challenge of modeling route choice behavior, bounded rationality adds more complexity due to the following reasons:

1. The existence of BR thresholds leads to mathematically intractable properties of equilibria.
2. BR parameters are usually latent and difficult to identify and estimate.
3. BR is associated with human being's cognitive process and is challenging to model.

3.1. Models

Simon (1986) classified two types of rationality: substantive rationality ('rationality is viewed in terms of the choices it produces') and procedural rationality (rationality is viewed 'in terms of the processes it employs'). Substantive rationality focuses on the choice results subjective to certain goals, while procedural rationality describes the cognitive process of a decision-maker. According to Simon (1982), bounded rationality is a more "ambitious" rationality concept, trying to capture both the substance of the final decision and the dynamical process of decision-making, based on empirical studies and psychological research. In route choice modeling, substantive bounded rationality aims to predict route choice outcomes, while procedural bounded rationality cares more about empirical studies of dynamic processes.

In terms of route choices, there also exist two categories of travel behavioral models: static traffic assignment (i.e., stable and time-invariant route choices) and dynamic traffic assignment (i.e., temporal travel behavioral changes with both spatial and temporal dimensions in the choice set). Both static and dynamic traffic assignment can be embedded into substantive and procedural bounded rationality.

To review BR related models and methodologies, a thorough survey on static and dynamic traffic assignment models should come along. Therefore, we will introduce various route choice models and show how substantive and procedural bounded rationality are represented.

3.2. Methodologies

The existing studies on boundedly rational route choice employ two diverging methodologies. Substantive bounded rationality focuses on modeling behavior with the game-theoretical approach and obtaining equilibrium link flows in a road network to facilitate transportation planning (i.e., normative theory of rational choice). Therefore bounded rationality parameters are assumed to be exogenous. Though substantive bounded rationality models can describe static and dynamic boundedly rational route choice behavior, the cognitive process leading to such behavior has not been fully explored. Therefore, procedural bounded rationality is proposed. Procedural bounded rationality aims to predict individuals' decision-making results and estimate bounded rationality parameters using the random utility model or non-/semi-compensatory strategies (i.e., positive theory of rational choice).

Before delving into the methodology framework, we want to pinpoint that aforementioned methodologists are equally important in boundedly rational route choice literature. Simon (1987) stated that bounded rationality theories are not simply "ad hoc and casual departures from the subjective expected utility theories underlying neoclassic economics", rather, the trademark attribute of contemporary bounded rationality theories should be their "detailed and systematic empirical study of human decision-making behavior in laboratory and real-world situations." According to this argument, any substantive models that only focus on the choice outcomes were not considered as hard-core "bounded rationality" theories. Even when bounded rationality is considered in the decision-making process (i.e., procedural rationality), the theories cannot be labeled contemporary "bounded rationality" theories without any empirical component. In transportation literature, however, outcomes, i.e., traffic equilibrium, is extensively studied, because of its critical role in long-term transportation planning. Solving traffic equilibrium will help estimate and predict traffic flows in a road network, which facilitates decision-makers for infrastructure investment plans. Therefore, in this paper, we will not only review bounded rationality associated with empirical studies, also dedicate a large amount of space in discussing traffic equilibrium with bounded rationality.

In decision theory, there exist two schools of models describing choice preference over uncertain outcomes: choice under certainty or choice under risk and uncertainty. The expected utility theory (i.e., normative or substantive model) assumes that decision-makers choose alternatives in terms of their expected utility, which deals with certainty. On the other hand, behavioral modeling under uncertainty takes into consideration decision-makers' risk-taking preference. Incorporating bounded rationality into expected utility models remains understudied, not mentioning those under uncertainty. In addition, sometimes not behaviorally rationally is rather a matter of complexity than a matter of uncertainty (Simon, 1972). For example, some choice, such as chess, is not inherently stochastic. It appears stochastic due to complexity of decision-making processes. In this paper, thus, we will mainly focus on reviewing boundedly rational behavioral modeling under certainty. In other words, expected utility or cost without risk is the main factor in making choices.

3.3. Summary of models

The traditional perfect rationality (PR) route choice paradigm (Wardrop, 1952) makes the following assumptions regarding human being's cognitive processes:

(1) Each traveler has access to information of all paths and their costs;
(2) Each traveler is able to enumerate all alternative paths connecting his or her origin-destination pair in a transportation network;
(3) Each traveler picks a path with the least disutility in static traffic assignment;
(4) Each traveler always switches to the path with the least disutility in dynamic route choice processes.

With these ideal assumptions, the detailed cognitive process modeling is dismissed in PR models. However, the above assumptions are too restrictive in reality (which are adapted from Simon, 1987), because:

(1) Accessing information of all paths is unrealistic because the costly information acquisition process prevents travelers from obtaining complete information of path costs. Many factors contribute to path costs, such as travel time, travel distance, the number of traffic lights and turns, weather, scenery, and so on. Some of these factors cannot be directly measured from the field or are difficult to measure;
(2) Due to the large size of available paths in real traffic networks and people's limited computational ability, it is impossible to identify all feasible paths connecting each origin-destination pair. In other words, "minimal completeness can seldom be guaranteed" (Simon, 1987);
(3) Human beings have limited cognitive capabilities such as "lack of knowledge and limited ability to forecast future" (Simon, 1987), which prevents them from acting as utility maximizers or disutitlity minimizers;
(4) Human beings have inertia, which prevents them from updating route information and switching routes too frequently.

Accordingly bounded rationality can be proposed to relax the classical PR models. Bounded rationality is a loosely defined term and different researchers incorporate it into different aspects. In the existing literature, bounded rationality is mainly represented in four aspects assuming (Simon, 1987):

(A) searching partial attributes information to obtain knowledge of path costs;
(B) considering a subset of feasible paths in choice set generation;
(C) (1) non-optimal route choice mechanism; (2) optimal route choice mechanism with perception errors;
(D) updating only non-salient information or switching to non-salient shorter paths in repeated route choice learning processes.

All relevant models are summarized in Table 1. Column "Category" represents two major types of bounded rationality aforementioned. Column "Aspect" divides each category based on whether time dimension is included. Column "Model" summarizes all the route choice models we will review in this paper. column "BR representation" further explains how the BR principle is incorporated (with the letter indicating the category of representation) while column "Parameter specification" explains the specific form of the associated bounded rationality parameters. Columns "Applications" and "References" list the context where the proposed models are applied to and their related references.

The survey of each category of models is arranged as follows: at the beginning of each section, a unifying framework diagram serves as a navigation map, providing a general picture of each category (including elements and their relations) and how BR pieces fit the whole picture. It is then followed by analytical models and/or estimation methodologies within the unifying framework.

4. Substantive bounded rationality: static game-theoretical models

Fig. 1 illustrates a framework of equilibrium models. A large population of travelers make route choices in a road network and suffer from congestion effects expressed in travel costs or disutilities. The travel cost or the disutility is indicated in

Table 1
Summary of boundedly rational travel behavior models.

Category	Aspect	Model	BR representation	Parameter specification	Applications	References
Substantive BR	Static traffic assignment (Section 4)	BRUE	Not take the shortest paths (C1)	An indifference band parameter varying among OD pairs	Flow equilibrium in disrupted network	Di et al. (2014; 2016; 2013); Guo (2013); Lou et al. (2010)
		IUE	Consider a subset of feasible paths (B)	Inertial path patterns among OD pairs	Drivers' compliance to information provision	Zhang and Yang (2015)
		SUE	Not choose the shortest paths due to perception errors and others' unknown choices (C2)	A rationality parameter varying among homogeneous users	More realistic flow equilibrium deviating from UE	Sheffi (1984)
		QRE	Not utility maximization due to perception errors and others' unknown choices (C2)	A rationality parameter varying among homogeneous/heterogeneous users	Finite-player congestion games with homogeneous or heterogeneous users	McKelvey and Palfrey (1995); Rogers et al. (2009)
		BRNE	Not utility maximization due to others' unknown choices (C2)	A rationality parameter varying among homogeneous users	Finite-player congestion games with subconscious utilities instead of utility functions	Chen et al. (1997)
	D2d/within-day traffic assignment (Section 5)	Deterministic route choice	Take an acceptable path every day (C1)	An indifference band parameter varying among OD pairs	Flow evolution prediction after a perturbation is imposed	Guo and Liu (2011)
		Stochastic route choice	Not always update perceived path costs (D)	An indifference band	Flow evolution prediction and parameter estimation after a perturbation is imposed	Wu et al. (2013)
		Dynamic traffic assignment	Not take the shortest paths at each time interval (C1)	A variable tolerance parameter depending on endogenous variables such as the departure rate vector	simultaneous route-and-departure-time choice evolution	Han et al. (2015); Szeto and Lo (2006)
		Dynamic congestion game	Not utility maximization due to others' unknown choices (C1)	A rationality parameter	Finite-player dynamic games	Chen et al. (1997); Han and Timmermans (2006); Zhao and Huang (2014)
Procedural BR	Static choice (Section 6)	Cognitive cost model	Consider a subset of alternatives via information search (A)	A random utility function in terms of search benefit and cost	Path information search given random path travel costs	Gao et al. (2011)
		Endogenous choice set generation	Consider a subset of alternatives via information search (A)	search gain v.s. search cost	A heuristic process of information search given random utilities	Richardson (1982)
		K-shortest path algorithms	Consider a subset of path attributes (B)	An indifference band parameter	exogenous choice set generation	Azevedo et al. (1993); de la Barra et al. (1993); Ben-Akiva et al. (1984); Zhu (2011)
		Non-compensatory heuristic	Consider a subset of attributes or paths (A,B)	An indifference band parameter	Exogenous choice set generation	Rasouli and Timmermans (2015)
		Reference-dependent model	Evaluate attributes/utilities to some reference point (C1)	An exogenous or endogenous reference point	Maximization of relative advantage, maximization of relative utility, minimization of regret	Rasouli and Timmermans (2015)

(*continued on next page*)

Table 1 (*continued*)

Category	Aspect	Model	BR representation	Parameter specification	Applications	References
		Minimum perceivable difference model	Indifferent to alternatives with small utility difference (C1)	A fixed parameter	Mode choice with addition of a new mode	Krishnan (1977); Lioukas (1984)
		Indifference to small changes in dynamic choices	Alternatives with a greater utility difference are considered with a higher probability (C1)	A random utility function with state dependence and/or serial correlation	Mode choice with addition of a new mode embedded with state dependence and serial correlation	Cantillo et al. (2007); 2006); Di et al. (2015b); 2015b)
		Sequential elimination by attributes	Consider a subset of attributes based on a prior ordering (A)	A tolerance parameter	Vehicle or destination choice with dynamic tolerance adjustment	Recker and Golob (1979)
		Elimination by aspects	Consider a subset of attributes and alternatives probabilistically (A)	The probability of examining one attribute is proportional to its importance	residential location choice	Tversky (1972); Young (1984)
		Semi-compensatory models	Consider a subset of alternatives by imposing constraints to attributes (A)	Attributes' cutoff points	Auto rental agency choice	Martínez et al. (2009); Swait (2001)
	Learning process (Section 7)	Travel time update	Only learn salient travel information (D)	An absolute indifference band	Travel time updating	Chen and Mahmassani (2004); Jha et al. (1998); Jotisankasa and Polak (2006); Nakayama et al. (2001)
		Departure-time and route choice	No switch unless schedule delay exceeds a threshold (D)	An absolute and a relative indifference bands	Departure-time and route switching	Chen and Mahmassani (2004); Hu and Mahmassani (1997); Jayakrishnan et al. (1994); Mahmassani and Chang (1987); Mahmassani and Jayakrishnan (1991); Mahmassani and Liu (1999); Srinivasan and Mahmassani (1999)
		Agent-based simulation	A positive search, information, learning, and knowledge theory (A)	A Bayesian learning process	Non-stationary route information search and choice	Zhang (2011)
		Stochastic automation	No departure-time switch unless schedule delay exceeds a threshold (D)	A fixed parameter	Departure-time choice adaptation	Yanmaz-Tuzel and Ozbay (2009)

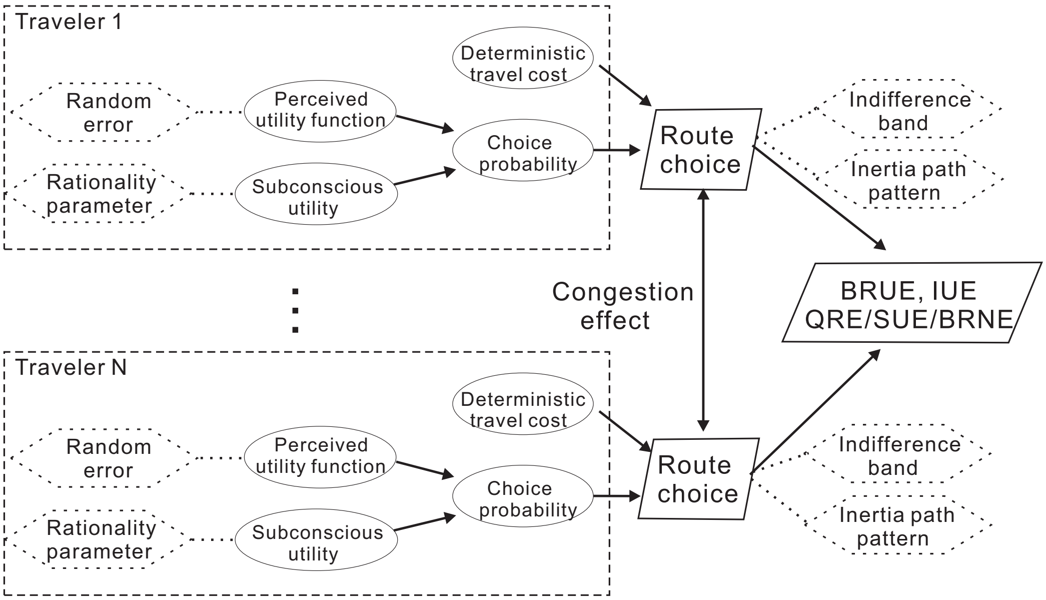


Fig. 1. Boundedly rational game-theoretical framework in route choice.

ovals which represent latent variables (matching symbols used by Walker, 2001) and is also enclosed in a big box with dotted borders because they are unobservable. Every traveler aims to minimize his or her own travel cost or disutility. Bounded rationality thresholds can be embedded into either cost or disutility functions in the form of "random error", "rationality parameter", or represented by the route choice principle, indicated in dotted hexagons. Individual's route choices and various types of equilibria, such as boundedly rational user equilibria (BRUE), initial user equilibrium (IUE), quantal response equilibrium (QRE), stochastic user equilibrium (SUE), and boundedly rational Nash equilibria (BRNE) are indicated in solid parallelograms, representing observable outputs. In this section, we will introduce how different specifications of travel costs or disutilities and different route choice behavior assumptions lead to different types of equilibria.

4.1. Static traffic assignment

In the game-theoretical framework, the traffic network is represented by a directed graph that includes a set of consecutively numbered nodes, $\mathcal{N}$, and a set of consecutively numbered links, $\mathcal{L}$. Let $\mathcal{W}$ denote the OD pair set and $w \in \mathcal{W}$ is connected by a set of simple paths (composed of a sequence of distinct nodes), $\mathcal{P}^w$, through the network. The traffic demand between each OD pair is d^w. $\mathbf{d}$ is a diagonal matrix which has wth diagonal element equal to demand d^w for OD pair w. Let f_r^w denote the flow on path $r \in \mathcal{P}^w$ for OD pair w, then the path flow vector is $\mathbf{f} = \{\mathbf{f}^w\}^{w\in\mathcal{W}} = \{f_r^w\}_{r\in\mathcal{P}^w}^{w\in\mathcal{W}}$. The feasible path flow set is to assign the traffic demand onto the feasible paths: $\mathcal{F} \triangleq \{\mathbf{f} : \mathbf{f} \geqslant \mathbf{0}, \sum_{r\in\mathcal{P}^w} f_r^w = d^w, \forall w \in \mathcal{W}\}$. Denote x_a as the link flow on link a, then the link flow vector is $\mathbf{x} = \{x_a\}_{a\in\mathcal{L}}$. Each link $a \in \mathcal{L}$ is assigned a cost function which is a function of link flows, written as $c(\mathbf{x})$. Let $\delta_{a,r}^w = 1$ if link a is on path r connecting OD pair w, and 0 if not; then $\Delta \triangleq \{\delta_{a,r}^w\}_{a\in\mathcal{L},r\in\mathcal{P}}^{w\in\mathcal{W}}$ denotes the link-path incidence matrix. So $\mathbf{x} = \Delta\mathbf{f}$. Denote $C_r^w(\mathbf{f})$ as the path cost on path r for OD pair w, then $C(\mathbf{f}) = \Delta^T\mathbf{c}(\mathbf{x})$ under the additive path cost assumption.

Wardrop's first principle says that people take the path with the least disutility (i.e., travel time, monetary cost, etc.). A Wardrop user equilibrium (UE) is reached when no one can improve his or her travel cost by unilaterally changing routes. At UE, all utilized routes have the minimum travel cost while all unused routes have higher travel costs. In other words, the following conditions hold at an equilibrium $\mathbf{f}^*$:

$$C_r^w(\mathbf{f}^*) - \pi^w \begin{cases} = 0, \text{ if } f_r^{w*} > 0, \\ \geqslant 0, \text{ if } f_r^{w*} = 0. \end{cases}, \forall r \in \mathcal{P}^w, \forall w \in \mathcal{W}, \tag{4.1}$$

where π^w is the minimum travel cost connecting OD pair w, i.e., $\pi^w = \min_{j\in\mathcal{P}^w} C_j^w(\mathbf{f})$.

4.1.1. Boundedly rational user equilibrium: non-optimal route choice

Under bounded rationality, at equilibrium, "every driver uses an acceptable path, where a path is acceptable if the difference between its travel cost and that of the shortest or least-cost path is no larger than a pre-specified threshold value" (Lou et al., 2010). In other words, no one can reduce his or her travel cost by a threshold by unilaterally switching routes. This threshold depends on network users' behavior and varies among different OD pairs, which needs to be obtained through behavioral surveys and experiments.

Given an indifference band, BRUE is shown to be non-unique (Lou et al., 2010). Lou et al. (2010) proposed a path-based BRUE formulation and a link-node based formulation for congestion pricing. The link-node based BRUE is shown to be more restrictive than the path-based one. Di et al. (2013) developed a nonlinear complimentary condition for one path-based BRUE flow pattern:

$$\tilde{C}_r^w(\mathbf{f}^*) - \tilde{\pi}^w \begin{cases} = 0, \text{ if } f_r^{w*} > 0, \\ \geqslant 0, \text{ if } f_r^{w*} = 0. \end{cases} \quad , \forall r \in \mathcal{P}^w, \forall w \in \mathcal{W}. \tag{4.2}$$

where,
$\tilde{C}_r^w(\mathbf{f}^*)$: the indifference travel cost, computed by $\tilde{C}_r^w(\mathbf{f}^*) = C_r^w(\mathbf{f}^*) + \rho_r^w$;
$\tilde{\pi}^w$: the highest acceptable path cost within the indifference band ε^w for OD pair w, equal to the minimum travel cost plus the indifference band connecting OD pair w, i.e., $\tilde{\pi}^w = \min_{j \in \mathcal{P}^w} C_j^w(\mathbf{f}) + \varepsilon^w$;

$\boldsymbol{\rho}$: an indifference vector and $\boldsymbol{\rho} = (\rho_r^w)_{r \in \mathcal{P}}^{w \in \mathcal{W}}$, where $0 \leqslant \rho_r^w \leqslant \varepsilon^w$. ρ_r^w represents the deviation of route r's actual cost from the shortest indifference travel cost, i.e., $\rho_r^w \triangleq \begin{cases} \tilde{\pi}^w - C_r^w(\mathbf{f}), & \text{if } C_r^w(\mathbf{f}) \leqslant \min_{j \in \mathcal{P}} C_j^w(\mathbf{f}) + \varepsilon^w, \\ 0, & \text{o.w.} \end{cases}$

The above condition implies that, under bounded rationality, all chosen paths have the same shortest indifference travel cost, equal to $\min_{j \in \mathcal{P}^w} C_j^w(\mathbf{f}) + \varepsilon^w$. The costs of the unused paths should be equal to or larger than the shortest indifference path cost.

Denote $\mathcal{F}_{BRUE}^{\boldsymbol{\varepsilon}}$ as the $\boldsymbol{\varepsilon}$-BRUE path flow solution set, including all the BRUE path flow patterns. Because it is generally a non-convex set (Lou et al., 2010), the BRUE path flow set is challenging to characterize. Di et al. (2013) developed a systematic methodology of constructing the BRUE set in transportation networks with fixed demands connecting multiple OD pairs. With the increase of the indifference band, the path set that contains boundedly rational equilibrium flows will be augmented. Accordingly, the critical values of indifference bands to augment these path sets can be identified by solving a family of mathematical programs with equilibrium constraints (MPEC) sequentially. For a network with single OD pair, given a sequence of finite critical points $\varepsilon_k^*, k = 1, \ldots, K$, with $\varepsilon_0^* = 0, \varepsilon_{K+1}^* = \infty$, a BRUE solution set is the union of $K+1$ subsets:

$$\mathcal{F}_{BRUE}^{\varepsilon} = \bigcup_{k=0}^{K} \mathcal{F}_k^{\varepsilon_k^*}. \tag{4.3}$$

where $\mathcal{F}_k^{\varepsilon_k^*}, k = 0, \ldots, K$ is the kth subset with associated critical indifference band ε_k^*.

Built upon BRUE solutions, the transportation network design problem under boundedly rational route choice behavior (BR-NDP) exhibit new features and bring in new challenges. Due to the existence of indifference bands, the road network may operate at different equilibrium flow patterns when new links are built or toll pricing is implemented, which results in uncertainty of road network operation. Accordingly, network planners may hold different attitudes, i.e., risk-averse, risk-prone, and risk-neutral, towards building a new link or charging tolls when uncertainty exists. Based on different attitudes towards risk, Di et al. (2014) investigated the occurrence of Braess paradox as both the travel demand and the indifference band vary. The Braess paradox under bounded rationality is completely new and Di et al. (2014) conducted the first analysis on risk-averse, risk-prone, and risk-neutral Braess paradox. Graphical analysis in the Braess network with affine link performance functions and numerical results in ordinary grid networks with regular BPR (Bureau of Public Roads) link performance functions were illustrated.

Due to convexity of the BRUE set, the existing BR related applications only focus on the continuous version of BR-NDP, BR toll pricing (abbreviated as "BR-TP"). To facilitate model formation and solution algorithms of BR-TP, the topological properties of the BRUE set, connectedness (Di et al., 2015a), compactness, non-convexity, and upper-semicontinuity (Di et al., 2016) were explored.

When there exist multiple equilibria in the lower level problem of BR-TP (i.e., each driver has multiple action strategies), planners do not know exactly how drivers will behave. There exists a gap between the "predicted" and the "realized" equilibria and solving a standard bi-level program may not achieve the expected goal of improving efficiency of the transportation system. Lou et al. (2010) formulated the risk-averse BR-TP based on more restrictive link-node BRUE formulation and proposed some heuristic algorithm to obtain a suboptimal toll. Furthermore, Di et al. (2016) modeled the risk-averse and risk-prone BR-TP based on BRUE path flow solutions and gave rigorous proofs of solution existence. The mathematical formulations of risk-averse and risk-prone BR-TP proposed in Di et al. (2016) are:

$$\begin{aligned} &\min_{\mathbf{y}} \; \min_{\mathbf{x}} / \max_{\mathbf{x}} \sum_{a \in \mathcal{A}} x_a t_a(x_a) + \sum_{a \in \mathcal{T}} x_a t_a(x_a, y_a) \\ &s.t. \qquad 0 \leqslant y_a \leqslant \bar{y}, a \in \mathcal{E} \\ &\qquad\qquad \mathbf{x} = \Delta \mathbf{f}, \forall \mathbf{f} \in \mathcal{F}_{BRUE}^{\varepsilon}(\mathbf{y}). \end{aligned} \tag{4.4}$$

where,
y_a: the toll charged on link a, between zero and an upper bound $\bar{y}$;
x_a: the link flow for link a;

$\mathcal{A}$: the link set without tolls charged;
$\mathcal{T}$: the link set with tolls charged;
$\mathbf{x} = \{x_a\}_{a\in\mathcal{X}}$: the link flow vector;
$t_a(x_a, y_a)$: the cost of traversing link a, which depends on total traffic using link a and the toll charged on link a;
$\mathcal{F}^{\varepsilon}_{BRUE}(\mathbf{y})$: the tolled BRUE solution set.

A decomposition algorithm paradigm taking advantage of the topological properties of BRUE sets was then proposed to solve risk-averse BR-TP in two numerical examples.

Guo (2013) proposed a totally different approach to stay away from the issue of non-uniqueness of BRUE. An iterative roll sequence operation was designed, which either forces one BRUE to converge to UE under homogeneous BR or guides flow to a reduced BRUE set or a subnetwork of UE under heterogeneous BR among different OD pairs.

4.1.2. Inertial user equilibrium: considering a subset of alternative paths

Zhang and Yang (2015) proposed another variant of user equilibrium based on a prevailing choice set of all the alternative routes, i.e., "initial user equilibrium (IUE)". At IUE, only a subset of feasible paths are considered. In addition, information provision may not be able to alter IUE due to the existence of inertia.

Define an inertia pattern H is to assign traffic demands between $O-D$ pair $w\in\mathcal{W}$ to a subset of feasible paths set $H\in\mathcal{P}^w$. Inertial user equilibrium follows Wardrops first principle in every inertia pattern $H\in\mathcal{H}^w$, where $\mathcal{H}^w$ is all the inertia patterns for OD pair $w\in\mathcal{W}$:

$$C_r^w(\mathbf{f}^*) - \pi^w \begin{cases} = 0, \text{ if } f_r^{w*} > 0, r\in H, \forall H\in\mathcal{H}^w \\ \geqslant 0, \text{ if } f_r^{w*} = 0, r\in H, \forall H\in\mathcal{P}^w\backslash\mathcal{H}^w. \end{cases}, \forall r\in\mathcal{P}^w, \forall w\in\mathcal{W}. \tag{4.5}$$

As the flow pattern defined by the above complementarity conditions may be non-unique, furthermore, a probability is assumed to be known over all inertia patterns connecting the same OD pair and thus: $\sum_{r\in H} f^w_{r,H} = d^w p^w_H$.

4.2. Congestion game: optimal route choice with perception errors

Non-cooperative game theory models finite players' strategic decisions in a conflict and competitive environment. In particular, in a congestion game (Rosenthal, 1973), the payoff of one player depends on the choice he or she chooses as well as the number of players choosing the same choice. Nash equilibrium is said to be achieved if no player can improve his or her payoff by changing strategies. Through repeated game experiments with finite players, researchers realize that the perfect Nash equilibrium cannot be usually obtained. To explain these anomalies, bounded rationality is then incorporated into non-cooperative mixed-strategy games in the form of sensitivity of payoff difference, inaccurate perception of payoff or cost functions. Rosenthal (1989) proposed that a player is insensitive to two choices if their payoff different is small. McKelvey and Palfrey (1995) assumed that game players are utility maximizers whose perception of utility functions is subject to noise. Chen et al. (1997) argued that players only know their subconscious utilities attached to each alternative instead of utility functional forms. The associated equilibrium is "t-solution" (Rosenthal, 1989), "quantal response equilibrium" (QRE) (McKelvey and Palfrey, 1995), and "boundedly rational Nash equilibrium" (BRNE) (Chen et al., 1997), respectively. The bounded rationality resides in the fact that inferior alternatives may be selected with positive but small probabilities. From players' perspectives, "better strategies are played more often than worse ones, but best strategies are not always played" (McKelvey and Palfrey, 1995). In the following, we will mainly introduce the seminal work proposed by McKelvey and Palfrey (1995) and its extension proposed in Zhao (1994). The bounded-rationality equilibrium proposed by Rosenthal (1989) can be treated as a linear version of QRE (McKelvey and Palfrey, 1995).

4.2.1. Quantal response equilibrium in congestion game

In a finite game $(\mathcal{M}, S, U)$, among $\mathcal{M} = \{1,\ldots,N\}$ total players, player n has a set of pure strategies $S^n = \left\{s_n^1,\ldots,s_n^{J_n}\right\}$. Each player knows the strategy sets available to both himself or herself and to others. The probability of player n taking strategy s_{nr} is $p_{nr}\triangleq p_n(s_{nr}) \geq 0$, and $\sum_{s_{nr}\in S^n} p_n(s_{nr}) = 1$. Denote $\mathbf{p}_n = \{p_{nr}\}^{r\in J_n}$ and $P = \{\mathbf{p}_n\}_{n\in\mathcal{M}}$.

Assume player n does not know other players' utility functions but can conjecture others' choice probability, which is called the "conjectured mixed strategy". Denote $p^{-n} = \{p_{n'r}\}^{r\in J_{n'}}_{n'\in\mathcal{M}, n'\neq n}$ as the conjectured mixed strategy adopted by players other than n and P^{-n} as all feasible mixed strategies other than player n. (s_{nr}, p^{-n}) is the strategy pair where player n adopts the pure strategy s_{nr} and conjectures that all other players adopt their components of p. Therefore, the expected utility of the rth pure strategy of player n, denoted as V_{nr}, is a function of the strategy pair, i.e., $V_{nr} = V_{nr}(s_{nr}, p^{-n})$. Denote the expected utility over all possible choices of all players other than n as $\bar{V}_{nr}$. Then,

$$\bar{V}_{nr} = \sum_{p^{-n}\in P^{-n}} \left(\prod_{n'\neq n} p_{n'r}(s_{n'r})\right) V_{nr}(s_{nr}, p^{-n}).$$

One example of the expected utility function $\bar{V}_{nr}$ is to assume that it depends on the total number of players choosing alternative r (Han and Timmermans, 2006), i.e.,

$$\bar{V}_{nr}(s_{nr}, p^{-n}) = \theta X_{nr}(s_{nr}) + f\left(N_r(p^{-n})\right),$$

where,
$X_{nr}(\cdot)$: a utility mapping only depending on player n's strategy s_{nr};
N_r: the expected number of players choosing alternative r, depending on others' choices;
$f(N_r(p^{-n}))$: a deterministic cost function depending on the expected number of players choosing alternative r.

Assume player n's utility for each strategy is subject to random error and is defined as:

$$U_{nr} = \bar{V}_{nr} + \zeta_{nr}, \tag{4.6}$$

where $\zeta_n = (\zeta_n^1, \ldots, \zeta_n^{J_n})$ is the perceived utility error vector for player n.

Player n will take strategy r if $U_{nr} \geqslant U_{nj}, \forall j = 1, \ldots, J_n, j \neq r$. The most common distribution of ζ follows i.i.d. Gumbel distribution with the scale parameter α. Its resultant equilibrium is logit equilibrium (LQRE) and the probability that player n selects strategy r is:

$$p_{nr} = P(U_{nr} > U_{nj}) = \frac{e^{\alpha \bar{V}_{nr}}}{\sum_{j=1}^{J_n} e^{\alpha \bar{V}_{nj}}}, j \neq r,$$

where α is the scale parameter for ζ_n^r. It also represents the rationality level of each player. When varying α from zero to infinity, the player's choice behavior varies from "placing equal probability over all alternatives" to "fully rational utility maximization" (McKelvey and Palfrey, 1995).

Every player has the same equilibrium conjecture of others' choices, i.e., $\mathbf{p}_n = \mathbf{p}_k = \mathbf{p}_*$. Therefore a statistical version of Nash equilibrium can then be defined as follows:

Definition 4.1. In a finite game $(\mathcal{M}, S, U)$, a quantal response equilibrium (QRE) is any $p = (p^1, \ldots, p^{J_n}) \in P$ such that $\forall n \in \mathcal{M}, 1 \leqslant r \leqslant J_n$,

$$p_*(\alpha) = \left\{ p \in P : p_{nr} = \frac{e^{\alpha \bar{V}_{nr}(p)}}{\sum_{j=1}^{J_n} e^{\alpha \bar{V}_{nj}(p)}}, \forall n, r \right\}. \tag{4.7}$$

McKelvey and Palfrey (1995) showed that QRE exists but is generally non-unique. However, it is unique when α is restricted to be sufficiently small. As α goes to infinity, there always exists one subsequence of $p_*(\alpha)$ which converges to a unique Nash equilibrium.

In QRE, players are assumed to be homogeneous, in other words, α and its resultant probabilities $p_*(\alpha)$ are the same across the entire population. By generalizing α to α_n for player n, Rogers et al. (2009) extended QRE to heterogeneous player types. The distribution of player n's type, denoted as $f_n(\alpha)$, is common knowledge to every player. However, each player does not know others' types. Given the prior probability of player n choosing strategy r (i.e., before α_n is drawn), i.e., $p_{nr}(\alpha)$, the induced mixed strategy is computed as: $\pi_{nr}(p) = \int_0^\infty p_{nr}(\alpha) f_n(\alpha) d\alpha$. Therefore a heterogeneous quantal response equilibrium (HQRE) is any $\pi = (\pi^1, \ldots, \pi^{J_n}) \in P$ such that $\forall n \in \mathcal{M}, 1 \leqslant r \leqslant J_n$,

$$\pi_*(\alpha) = \left\{ \pi \in P : \pi_{nr} = \frac{e^{\alpha_n \bar{V}_{nr}(\pi(p))}}{\sum_{j=1}^{J_n} e^{\alpha_n \bar{V}_{nj}(\pi(p))}}, \forall n, r \right\}. \tag{4.8}$$

QRE is a set of mixed strategies each individual player should follow while playing the game. For example, in a road network with two routes and each route has the equal probability of being selected, one traveler will first flip a coin and then choose the first route if it lands with head and the second if it lands with tail. In traffic assignment, we are more interested in the aggregate number of users choosing each route rather than individual choice probability. As LQRE or HQRE is a finite-player game, meaning one user cannot be divided into "infinitesimal" fractions, the numbers of users on each path are assumed to follow a multinomial distribution with $d^w, \forall w \in \mathcal{W}$ as the number of trails and $p_r, \forall r \in \mathcal{P}$ as event probabilities. One random draw from the multinomial distribution gives one realization of the equilibrium choosing each alternative based on LQRE.

4.2.2. Stochastic user equilibrium in continuous congestion game

Continuous congestion game is the limiting case of congestion game as the number of players goes to infinity. Accordingly, in the route choice context, Nash equilibrium converges to Wardrop user equilibrium (Haurie and Marcotte, 1985). Wardrop user equilibrium, with infinitesimal travelers in a non-cooperative Nash game, is reached when no traveler can improve his or her travel cost by unilaterally switching routes.

There exists a branch of studies on Nash equilibrium embedded with bounded rationality, but there does not exist many studies on boundedly rational Wardrop user equilibrium. Stochastic user equilibrium (SUE) is one commonly used equilibrium with the boundedly rational principle.

Assume each path's travel cost is random due to perception error, i.e., $C_r^w = \bar{C}_r^w(\mathbf{f}) + \zeta_r^w$, where $\bar{C}_r^w$ is the deterministic flow-dependent perceived travel cost and ζ_r^w represents the perceived random error. At SUE, no traveler can improve his or her perceived travel time by unilaterally switching routes:

$$\bar{C}_r^w(\mathbf{f}^*) - \pi^w \begin{cases} = 0, \text{ if } f_r^{w*} > 0, \\ \geqslant 0, \text{ if } f_r^{w*} = 0. \end{cases}, \forall r \in \mathcal{P}^w, \forall w \in \mathcal{W}, \tag{4.9}$$

where

$$p_r^w = \frac{\exp(-\alpha \bar{C}_r^w)}{\sum_{j \in \mathcal{P}^w} \exp(-\alpha \bar{C}_j^w)}, \tag{4.10a}$$

$$f_r^w = d^w p_r^w. \tag{4.10b}$$

When $\alpha = 0$, each route has the same probability of being chosen; when $\alpha \to \infty$, SUE converges to UE. SUE can be formulated as a convex optimization problem and its uniqueness can be guaranteed when the link cost function is separable and monotone increasing.

Remark.

1. Both SUE and QRE are derived from the statistical setting of game theory. SUE gives a set of expected number of users for each route and QRE is a set of mixed strategies each individual player should follow while playing the game. Therefore QRE is derived from each individual player's response strategy, while SUE is focused on an aggregate level of traffic flow. Loosely speaking, if each player has the same payoff function and the same mixed strategy, in continuous congestion game, choice probabilities can be interpreted as proportions. Consequently the expected number of players choosing each alternative should be the product of the total number of players and its choice probability. In this sense, SUE is the expectation of equilibrium based on LQRE with homogeneous players shown in Eq. (4.10) in continuous congestion game. Though their mathematical results turn to the same, but rigorous proof of their equivalence is further needed and will be left for future research.
2. We also need to highlight that in discrete game, LQRE may be different from SUE. As SUE allows fractional users on each route, while LQRE describes finite players. We conjecture that LQRE may be converged to SUE in large population approximation but such a claim needs to be mathematically proved.
3. There exists a minus sign before α in Eq. (4.10a), but no minus sign in Eq. (4.7). The reason is, SUE is defined based on disutility, i.e., travel time, while QRE is defined on utility. However their formulations are essentially equivalent. To remove the minus sign in Eq. (4.10a), we can simply add a minus sign in front of travel time to convert it to utility.

4.2.3. Boundedly rational Nash equilibrium

Chen et al. (1997) argued that the mathematical interpretations of choice behavior by introducing noise into the utility function cannot manifest human being's bounded rationality. Accordingly, a boundedly rational Nash equilibrium model (BRNE) is developed based on the assumption that the player does not know his utility function, instead, he or she knows utility values associated with each alternative, v_n^r. This latent utility is called "subconscious utility" and the subconscious utility of the rth pure strategy of player n given others' choice probability p^{-n} can be computed as: $\bar{V}_{nr}(s_{nr}, p^{-n}) \triangleq \bar{V}_{nr}(P^n) = \sum_{p^{-n} \in P^{-n}} (\prod_{n' \neq n} p_{n'r}(s_{n'r})) v_{nr}$. The probability of choosing one pure strategy r is defined as:

$$p_{nr}(P^n) = \frac{f(\bar{V}_{nr}(P^n), \alpha_n)}{\sum_{j=1}^{J_n} f(\bar{V}_{nj}(P^n), \alpha_n)},$$

where α_n is the rationality parameter of player n and $f(\cdot)$ is a generic function of the expected subconscious utility.

Definition 4.2. A mixed strategy profile $\pi = (\pi^1, \ldots, \pi^{J_N}) \in P$ is a boundedly rational Nash equilibrium (BRNE) if $\forall n \in \mathcal{M}, r = 1, \ldots, J_n$,

$$\pi_*(\alpha) = \left\{ \pi \in P : \pi_{nr} \triangleq p_{nr}\left(P^n(\pi^{-n})\right) = \frac{f\left(\bar{V}_{nr}(P^n(\pi^{-n})), \alpha_n\right)}{\sum_{j=1}^{J_n} f\left(\bar{V}_{nj}(P^n(\pi^{-n})), \alpha_n\right)}, \forall n, r \right\}. \tag{4.11}$$

When $f\left(\bar{V}_n^r(P^n(\pi^{-n})), \alpha_n\right) = \exp\left(\alpha^n \bar{V}_n^r(P^n(\pi^{-n}))\right)$, BRNE is mathematically equivalent to the LQRE with heterogeneous players. However, we need to point out that QRE and BRNE are developed based on different behavioral assumptions.

4.3. Relations of boundedly rational equilibria

To briefly illustrate the relationships between a variety of bounded rationality associated equilibria reviewed in the previous sections, we will compute aforementioned equilibria in the Braess paradox network.

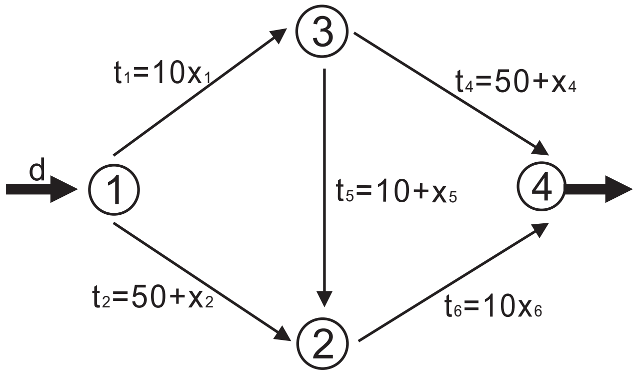


Fig. 2. The Braess network.

Example 4.1. The topology of the network, the OD demand between nodes $1-4$, and link cost functions are illustrated in Fig. 2. There are three paths: 1-3-4 (path 1), 1-2-4 (path 2), and 1-3-2-4 (path 3). The total demand $d=4$ and the indifference band $\varepsilon=3$.

For the fixed demand level, due to flow conservation, the feasible flow set can be characterized by two path flows, i.e., $f_3=d-f_1-f_2$. Therefore, three path costs can be computed as functions of flows on the first two paths:

$$\begin{aligned} C_1 &= 11f_1+10f_3+50 = f_1-10f_2+90,\\ C_2 &= 11f_2+10f_3+50 = -10f_1+f_2+90,\\ C_3 &= 10(f_1+f_2)+21f_3+10 = -11(f_1+f_2)+94. \end{aligned}$$

UE is computed by letting three path costs equal, i.e., $C_1=C_2=C_3$, and the flow pattern needs to be feasible, i.e., $f_1+f_2+f_3=4,\ f_1,f_2,f_3\geqslant 0$. Then we have

$$\mathbf{f}_{UE}=\begin{bmatrix}\frac{4}{13} & \frac{4}{13} & \frac{44}{13}\end{bmatrix}^T. \tag{4.12}$$

The BRUE set is composed of only one set: $\mathcal{F}^{\varepsilon=10}_{BRUE}=\mathcal{F}^{\varepsilon=10}_{0}$. All three paths are utilized under UE. Therefore under BRUE these three paths are all acceptable. According to Di et al. (2013), the BRUE set can be represented as:

$$\begin{aligned} \mathcal{F}^{\varepsilon}_{BRUE}=\{\mathbf{f}\in\mathcal{F}: & f_1+f_2+f_3=4,\ f_1,f_2,f_3\geqslant 0,\\ & |f_1-f_2|\leqslant\frac{\varepsilon}{11};\\ & |12f_1+f_2-4|\leqslant\varepsilon;\\ & |f_1+12f_2-4|\leqslant\varepsilon\}. \end{aligned}$$

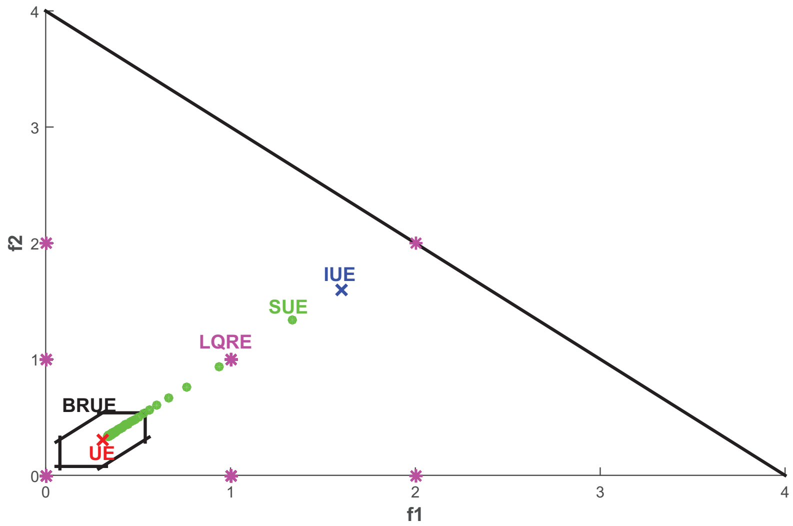
$$\mathbf{f}_{SUE}(\alpha)\begin{cases} f_1=4\dfrac{e^{-\alpha(C_1)}}{e^{-\alpha(C_1)}+e^{-\alpha(C_2)}+e^{-\alpha(C_3)}},\\ f_2=4\dfrac{e^{-\alpha C_2}}{e^{-\alpha(C_1)}+e^{-\alpha(C_2)}+e^{-\alpha(C_3)}},\\ f_3=4\dfrac{e^{-\alpha C_3}}{e^{-\alpha(C_1)}+e^{-\alpha(C_2)}+e^{-\alpha(C_3)}}. \end{cases} \tag{4.13}$$

In simulation, we set α ranging from 0 to 6 with a stepsize of 0.1.

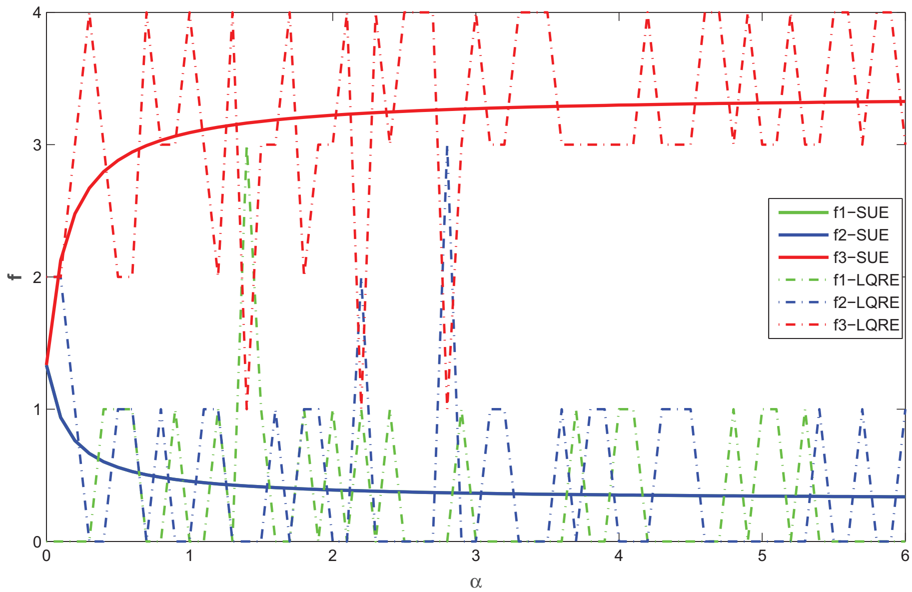
Assume four users are homogeneous, in other words, they have the same probability of choosing a particular path. To calculate LQRE, we need to know the expected utility $\bar{V}^r_n,\ \forall r,n$ in Equation (2.4.1). In this example, the expected disutility is the expected travel cost $\bar{C}^n_r,\ r=1,2,3,\ n=1,2,3,4$. As four users experience the same travel cost for the same path, we can omit the superscript n.

Because the number of users on each path follows a multinomial distribution, the expected flow is equal to the total demand times the choice probability, i.e., $\bar{f}_r=dp_r,\ r=1,2,3$. Then the expected path costs can be computed as:

$$\begin{aligned} \bar{C}_1 &= \bar{f}_1-10\bar{f}_2+90 = dp_1-10dp_2+90,\\ \bar{C}_2 &= -10\bar{f}_1+\bar{f}_2+90 = -10dp_1+dp_2+90,\\ \bar{C}_3 &= -11(\bar{f}_1+\bar{f}_2)+94 = -11(dp_1+dp_2)+94. \end{aligned}$$



(a) Flows on the first two paths at equilibria



(b) Path flows versus α for LQRE and SUE

Fig. 3. Illustration of boundedly rational equilibria.

For LQRE, the mixed strategy is:

$$\mathbf{p}_{LQRE}(\alpha)\begin{cases} p_1 = \dfrac{e^{-\alpha\bar{C}_1}}{e^{-\alpha\bar{C}_1}+e^{-\alpha\bar{C}_2}+e^{-\alpha\bar{C}_3}}, \\ p_2 = \dfrac{e^{-\alpha\bar{C}_2}}{e^{-\alpha\bar{C}_1}+e^{-\alpha\bar{C}_2}+e^{-\alpha\bar{C}_3}}, \\ p_3 = \dfrac{e^{-\alpha\bar{C}_3}}{e^{-\alpha\bar{C}_1}+e^{-\alpha\bar{C}_2}+e^{-\alpha\bar{C}_3}}. \end{cases} \tag{4.14}$$

BRNE is the same as LQRE if multinomial logit function is used. So we will omit calculation of BRNE.
From Fig. 3a, we have the following conclusions:

1. SUE converges to UE as α grows;
2. UE is contained in the BRUE set;
3. The BRUE set in this example is convex;

4. In BRUE and LQRE, path 3 is used while in other equilibria, only path 1 and 2 are used;
5. All other equilibria are unique except BRUE.

Fig. 3b illustrates the traffic flows on each path for SUE and LQRE. To establish their connections, we will use symbols for the purpose of generalization. Multiply the total demand d on both sides of Eq. (4.14), we get:

$$\mathbf{f}_{LQRE}(\alpha)\begin{cases} dp_1 = \bar{f}_1 = d\dfrac{e^{-\alpha\bar{C}_1}}{e^{-\alpha\bar{C}_1}+e^{-\alpha\bar{C}_2}+e^{-\alpha\bar{C}_3}}, \\ dp_2 = \bar{f}_2 = d\dfrac{e^{-\alpha\bar{C}_2}}{e^{-\alpha\bar{C}_1}+e^{-\alpha\bar{C}_2}+e^{-\alpha\bar{C}_3}}, \\ dp_3 = \bar{f}_3 = d\dfrac{e^{-\alpha\bar{C}_3}}{e^{-\alpha\bar{C}_1}+e^{-\alpha\bar{C}_2}+e^{-\alpha\bar{C}_3}}. \end{cases} \tag{4.15}$$

By comparing Eqs. (4.13) and (4.15), we can clearly see that the expected number of travelers on each paths for LQRE is the same as SUE. However, for one round, players will randomize their path choices and choose paths based on the mixed strategy defined in Eq. (4.14). That is why LQRE fluctuates around SUE. As four players are small, LQRE and SUE diverge due to randomization. In other words, SUE can be fraction numbers while LQRE need to be integers. We can thus suspect that as the total number of players goes to infinity, homogeneous LQRE will converge to SUE.

5. Substantive bounded rationality: day-to-day and within-day traffic assignment

Dynamical travel choices can be studied on a daily basis or within a day. According to different time scales, a dynamic process can be represented by a day-to-day model or a within-day model. In this section, we will first review analytical models of day-to-day and within-day dynamics and then introduce parameter estimation for each stage.

5.1. Day-to-day traffic assignment

Day-to-day traffic dynamical systems model drivers' route choice adjustment in response to temporary changes of a traffic network based on previous experienced travel costs. There are two classes of traffic dynamics in the existing literature: (1) deterministic user equilibrium dynamical systems (Friesz et al., 1994; He et al., 2010; Nagurney and Zhang, 1997; Smith, 1979), adopting various route choice update mechanism, such as proportional-switch adjustment (Smith, 1979), tatonnement adjusting process (Friesz et al., 1994), dynamical projection (Nagurney and Zhang, 1997) or link-based adjustment (He et al., 2010); and (2) stochastic day-to-day dynamics (Cascetta, 1989; Davis and Nihan, 1993; Watling, 1999), assuming drivers follow logit or probit model. Provided certain regulation conditions, these dynamical systems converge to different types of equilibria: the deterministic user equilibrium dynamical systems stabilizes to user equilibrium (UE) and the stochastic day-to-day dynamics' equilibrium is characterized by a stationary distribution with stochastic user equilibrium (SUE) as its mean. In this section, we will discuss how BR is embedded into these two types of traffic dynamics.

5.1.1. Deterministic day-to-day dynamic

The adjustment processes (Friesz et al., 1994; He et al., 2010; Nagurney and Zhang, 1997) assume: on each day, the flow pattern tends to move from the current pattern $\mathbf{f}$ towards the target pattern $\mathbf{u}$, based on current day's path costs $C(\mathbf{f})$ or link costs $c(\mathbf{x})$ (Friesz et al., 1994; He et al., 2010; Nagurney and Zhang, 1997). He (2010) proposed a general framework of existing day-to-day dynamics:

$$\mathbf{f}^t = \mathbf{f}^{t-1} + \lambda(\mathbf{u}^t - \mathbf{f}^t), \tag{5.1a}$$

$$\mathbf{u}^t = \Pr_{\Omega}(\mathbf{f}^t - \gamma C(\mathbf{f}^t)), \tag{5.1b}$$

where,
$\mathbf{f}$: the path or the link flow vector;
$\mathbf{u}$: target flow pattern;
λ: a positive constant determining the flow changing rate;
$\mathbf{u}-\mathbf{f}$: a flow changing direction;
Ω: the feasible path or link flow set, can be either $\mathcal{P}$ or $\mathcal{X}$;
$\Pr_{\Omega}(\mathbf{f}^t - \gamma C(\mathbf{f}^t))$: projection operator, projecting $\mathbf{f}-\gamma C(\mathbf{f})$ onto Ω, where γ is a coefficient.

The aforementioned day-to-day dynamics mainly focus on traffic evolution from disequilibrium to equilibrium within a network with fixed topology. When the topology of a network is changed, such as a disrupted or restored network, travelers may behave differently from when the network is stable and the existing perfectly rational day-to-day dynamics will not work. He and Liu (2012) proposed a prediction–correction process to describe travelers' reaction within a disrupted network. Guo and Liu (2011) developed a boundedly rational route choice dynamic to capture travelers' route choices in face of a restored network. As the dynamic proposed by Guo and Liu (2011) involves bounded rationality, we will briefly discuss this model.

Under certain regularity assumptions (Cantarella and Cascetta, 1995), the existing perfectly rational day-to-day dynamic has an unique fixed point. Accordingly the dynamic converges to the same UE flow pattern if a network first disrupts and is then restored to the original level. This cannot capture irreversible response to the network change observed in Minnesota (Di et al., 2015b; Guo and Liu, 2011). Therefore perfectly rational day-to-day dynamics will predict the same amount of traffic flow across the bridge. By allowing drivers' perception errors to vary in a presumed bound, a link-based boundedly rational day-to-day dynamic (Guo and Liu, 2011) successfully explained the flow reduction phenomenon whereas traffic evolves from one fixed point towards another within the BRUE solution set.

In the link-based boundedly rational day-to-day dynamic (Guo and Liu, 2011), Eq. (5.1b) is replaced with the following:

$$\mathbf{u} = \Pr_{\Omega^{br}(c(\mathbf{x}))}(\mathbf{x}), \tag{5.2}$$

where $\mathbf{x}$ is the link flow and $\Omega^{br}(c(\mathbf{x}))$ is the acceptable link flow pattern induced by $\mathbf{x}$.

Before defining $\Omega^{br}(c(\mathbf{x}))$, we need to first define the acceptable path set induced by link cost $c(\mathbf{x})$:

$$\mathcal{P}^{br}(c(\mathbf{x})) \triangleq \{r \in \mathcal{P} : C_r(\mathbf{f}) \leqslant \min_{j \in \mathcal{P}^w} C_j(\mathbf{f}) + \varepsilon\}. \tag{5.3}$$

where $\mathbf{f}$ is the path flow vector.

According to (5.3), $\mathcal{P}^{br}(c(\mathbf{x}))$ can be computed in the following steps: based on the current link cost $c(\mathbf{x})$, the path cost $C(\mathbf{f})$ can be calculated for each OD pair. Find the shortest path, the 2nd shortest path, ..., until the rth shortest path which has the cost difference from the shortest one less than ε, while the $(r+1)$th shortest path with the cost difference from the shortest one greater than ε. Then these r paths are acceptable paths.

After the acceptable path set $\mathcal{P}^{br}(c(\mathbf{x}))$ is known, assign the demand to those acceptable paths for each OD pair, then $\Omega^{br}(c(\mathbf{x}))$ can be mathematically expressed as:

$$\Omega^{br}(c(\mathbf{x})) \triangleq \left\{ \mathbf{x} \in \mathcal{X} : \mathbf{x} = \Delta \mathbf{f}, \sum_{r \in \mathcal{P}^{br}(c(\mathbf{x}))} f_r^w = d^w, \forall w \in \mathcal{W} \right\}. \tag{5.4}$$

The fixed point of this dynamic is the BRUE instead of the unique UE. Therefore the stability property of this dynamic is more difficult to address. Its stability is defined as: the perturbation of a fixed point will make the system to converge to a fixed point within the set. The new fixed point can be the same as or different from the initial one. Rigorous proofs of the stability property of the BR dynamic were presented in Di et al. (2015a).

5.1.2. Stochastic day-to-day dynamic

The existing deterministic day-to-day dynamics is a tool in modeling path choices based on the previous day's experience, while the existing stochastic day-to-day dynamics have the capability in both modeling and parameter estimation. In this section, we will focus on modeling. Parameter estimation of stochastic day-to-day dynamics will be introduced in Section 7.1.

Denote the expected state vector of the stochastic day-to-day dynamic as $\begin{bmatrix}\mathbf{g}^t \\ \mathbf{f}^t\end{bmatrix}$. Then the expected states of a stochastic day-to-day dynamic can be defined in a compact form:

$$\begin{bmatrix}\mathbf{g}^t \\ \mathbf{f}^t\end{bmatrix} = h\left(\begin{bmatrix}\mathbf{g}^{t-1} \\ \mathbf{f}^{t-1}\end{bmatrix}\right), \tag{5.5}$$

where,

$h(\cdot)$: a nonlinear mapping which will be specified later;

$\mathbf{g}^t$: the perceived travel cost vector on day t;

$\mathbf{f}^t$: the path flow on day t.

The mapping $h(\cdot)$ defines a nonlinear Markov process. It is continuously differentiable with respect to the state if link cost functions are continuous. If all eigenvalues of the Jacobian matrix of $h(\cdot)$ are within a unit circle (Cantarella and Cascetta, 1995), the dynamic will converge to a stationary flow distribution.

The mapping $h(\cdot)$ can be further specified according to two stages defined by Cantarella and Cascetta (1995). Cantarella and Cascetta (1995) built a unifying theory of both day-to-day and within-day traffic dynamics, including learning/forecasting mechanism and users' choice behavior. In the following, we will introduce how to define $h(\cdot)$ in these two stages.

Travelers update their perceived travel costs based on a weighted average of the previous day's perceived travel cost and the experienced cost (Cantarella and Cascetta, 1995). Mathematically,

$$\mathbf{g}^t = \lambda \mathbf{g}^{t-1} + (1-\lambda) C(\mathbf{f}^{t-1}), 0 < \lambda < 1. \tag{5.6}$$

where λ is a constant parameter between 0 and 1.

In reality, travelers may be salient to the travel cost difference between the previous day's perceived travel cost and the experienced cost if its value is within a threshold. Therefore, Wu et al. (2013) revised the cost update mechanism defined in Eq. (5.6) as follows:

$$\mathbf{g}^t = \begin{cases} \lambda \mathbf{g}^{t-1} + (1-\lambda) C(\mathbf{f}^{t-1}), & \text{if } |\mathbf{g}^{t-1} - C(\mathbf{f}^{t-1})| \geqslant \varepsilon, \\ g^{t-1}, & \text{o.w.} \end{cases} \tag{5.7}$$

This travel time updating along with a logit route choice model is applied to model travelers' day-to-day evolution within urban railway networks and this updating model better captures the day-to-day dynamic (Wu et al., 2013). Due to boundedly rational cost updating, this dynamic is not continuously differentiable with respect to its state any more. Its convergence and stability properties, unaddressed by Wu et al. (2013), are more challenging to identify and remain unanswered.

At the second stage of users' choice behavioral modeling, the expected path choice dynamic can be defined as a logit model $\mathbf{f}^t = \mathbf{d}P(\mathbf{g}^t)$, where $P(\cdot)$ is a logit probability. More generally (Cantarella and Cascetta, 1995),

$$\mathbf{f}^t = P(\mathbf{g}^{t-1}, C(\mathbf{f}^{t-1}))\mathbf{f}^{t-1}, \tag{5.8}$$

where $P(\cdot)$ is a transition probability matrix, depending on previous day's perceived and actual costs.

The transition matrix $P(\cdot)$ can be further decomposed into two parts (Cantarella and Cascetta, 1995): reconsidering previous day's choice and choosing today's path. Then the probability of choosing a path is the probability of reconsidering the previous day's choice (switching choice) times the conditional probability of choosing that path given that the previous day's choice is reconsidered (path choice). BR can be embedded into calculating both switching choice and path choice. The representation of BR in these two choices along with the methodology of estimating parameters will be introduced in Section 7.1.

5.2. Dynamic traffic assignment

Dynamic traffic assignment (DTA) models traveler's temporal travel choice change. Peeta and Ziliaskopoulos (2001) classified existing dynamic traffic assignment models into four methodological groups: discrete-time mathematical programming, continuous-time optimal control, variational inequality and simulation-based. To the best of our knowledge, bounded rationality has been only embedded into the third and the forth categories in existing literature, which will be explained in the following.

5.2.1. Variational inequality

A vector $\mathbf{f} \in \mathcal{F}$ is a dynamic user equilibrium (DUE) if and only if it solves an infinite dimensional variational inequality (VI) $\forall w \in \mathcal{W}$:

$$\sum_{r \in \mathcal{P}^w} \int_{t_0}^{t_f} C_r^w(t, \mathbf{f}^*)(f_r - f_r^*)dt \geqslant 0. \tag{5.9}$$

Szeto and Lo (2006) indicated that a DTA model with a physical queue paradigm may not even converge. To solve this anomaly, a tolerance-based principle was proposed that a route carrying flow at a time interval has a travel cost which is less than an acceptable tolerance from the shortest travel cost. As an extension of the tolerance-based model, Han et al. (2015) proposed a boundedly rational simultaneous route-and-departure choice dynamic, which is more mathematically elegant and rigorous and thus will be our main focus.

Under bounded rationality, a boundedly rational dynamic user equilibrium (BR-DUE) is reached when "travel times of all used routes between the same OD pair are within an acceptable tolerance from the minimum OD route travel time" at each time interval (Szeto and Lo, 2006). Given $\{\varepsilon^w\}^{w\in\mathcal{W}}$, a BR-DUE satisfies the following variational inequality:

$$\sum_{r \in \mathcal{P}^w} \int_{t_0}^{t_f} \phi_r^\varepsilon(t, \mathbf{f}^*)(f_r - f_r^*)dt \geqslant 0, \tag{5.10}$$

where $\phi_r^\varepsilon(t, f) \triangleq \min\{C_r(t, \mathbf{f}), \min_{j\in\mathcal{P}^w} C_j(t, \mathbf{f}) + \varepsilon^w\}$, denoting the minimum of the actual cost and the minimum cost plus the indifference band.

The rationale underlying VI formulation can be interpreted as follows: assume there is only one path r, to find BR-DUE $\mathbf{f}^*(t)$, it requires whenever $f_r^*(t)$ is non-zero, $C_r(t, \mathbf{f}) \leqslant \min_{j\in\mathcal{P}^w} C_j(t, \mathbf{f}) + \varepsilon^w$. In other words, $f_r^*(t) = \arg\min_{\mathbf{f}\in\mathcal{F}} \int_{t_0}^{t_f} \phi_r^\varepsilon(t, \mathbf{f}^*) f_r dt$ and the left-hand side of Eq. (5.10) should be zero at BR-DUE.

Han et al. (2015) pointed out that the existence of DUE requires the path cost $C(t, \mathbf{f})$ to be Lipschitz continuous with respect to both t and $\mathbf{f}$. However, to ensure the existence of BR-DUE, $C(t, \mathbf{f})$ only needs to be Lipschitz continuous with respect to $t, \forall \mathbf{f} \in \mathcal{F}$. Therefore the existence of BR-DUE is weaker than that of DUE. In other words, under certain conditions, DUE may not exist but BR-DUE still exists and is not unique. Another contribution of this paper was to characterize the solution set of discrete time BR-DUE. The solution set is shown to be compact and there always exists a connected but possibly nonconvex subset which can be generated from a single solution. Three algorithms were proposed to solve BR-DUE along with rigorous convergence results: a fixed-point method, a self-adaptive projection method, and a proximal point method.

5.2.2. Simulation

Various DTA simulators, such as DYNASMART (Jayakrishnan et al., 1994), DynaMIT (Ben-Akiva et al., 1997), and RouteSim (Ziliaskopoulos and Waller, 2000), are employed to compute DUE. Simulation cannot obtain analytical properties of DUE but it satisfies FIFO (First In First Out) and circumvents holding-back of vehicles. DTA simulators usually include three components (Cantarella and Cascetta, 1995): demand/supply model, learning/forecasting mechanism, and choice behavior.

BR principle can be introduced into learning/forecasting mechanism and choice models in DTA simulators, which is achieved by the simulator: DynusT (Chiu et al., 2011). DynusT (Chiu et al., 2011), a window-based open source platform, estimates dynamic traffic flow evolution patterns in large networks based on dynamic equilibrium assignment. It is widely used by state agencies for transportation policy decision making. One user class, named "En-Route Info", is introduced in simulation to represent those travelers who behave boundedly rational under the provision of real-time information. Users are allowed to input two boundedly rational thresholds into the simulator: (a) the indifference band representing people's inertia for switching to a new route, and (b) the threshold bound representing the difference between the chosen travel time and recommended travel time.

5.3. Dynamical congestion game and belief learning

There are two types of learning (Han and Timmermans, 2006):

1. Belief learning: learn from what choices others have taken in the past;
2. Reinforcement learning: learn from previous experiences.

Belief learning is used in the game-theoretical approach, while reinforcement learning is primarily used in random utility models. In this section, we will discuss belief learning, while reinforcement learning will be introduced in Section 7.

When a congestion game is played repeatedly, Chen et al. (1997) proposed a boundedly rational dynamical game and established the connection between static and dynamic game. Supposing that the subconscious utility a player has depends on that player's beliefs about others' strategies, when the player repeatedly play the game, this belief can be obtained by the observed choices of other players in the past. Eventually the stead-state distributions of every player's decisions will be learned by all players. So a dynamical adjustment process using fictitious play is modeled where each player plays the game based on others' historic strategies. Define the empirical distribution of player n is $\bar{P}_{nt} = (\bar{P}^1_{nt}, \ldots, \bar{P}^{J_n}_{nt})$, which can be calculated as:

$$\bar{P}^t_n = \frac{1}{t}\sum_{\tau=0}^{t-1} \hat{p}^{\tau}_n. \tag{5.11}$$

where $\hat{p}_{n\tau}$ is the actual choice probability of player n at time τ.

Based on empirical distributions, the subjective utilities can be calculated every day prior to taking action. It is shown that both players' beliefs about others' strategies and actual choices converge in probability to a boundedly rational Nash equilibrium (BRNE).

Han and Timmermans (2006) integrated both belief and reinforcement learning in departure time choice to test how well the proposed learning mechanism converges to QRE:

$$I^t_n = \lambda I^{t-1}_n + 1, \tag{5.12a}$$

$$V^t_{nr} = \frac{\lambda I^{t-1} V^{t-1}_{nr} + \bar{V}_{nr}(s_{nr}, p^{-n})}{\lambda I^{t-1}_n + 1}, \tag{5.12b}$$

where,
λ: a constant parameter between 0 and 1.
V^t_{nr}: utility belief of alternative r from traveler n.
I^t_n: an indicator of past experience relative to current experience. To simulate decline learning effect, $I^{t-1}_n \leqslant I^t_n$.

Numerical simulation showed that the low initial experience weight and the high learning parameter expedite learning process and enhance dynamic convergence. The learning process will eventually converge to QRE.

Zhao and Huang (2014) borrowed the QRE framework to model route choice behavior in a small network with one OD pair connecting two parallel links. Travelers are assumed to become more rational (i.e., α increases) through repeated game play, therefore an exponential learning curve is defined: $\alpha(t) = \alpha e^{\lambda(t-1)}$ where λ is the rate of learning. It showed that QRE converges to Nash equilibrium (also UE) as the rationality parameter α goes to ∞ or as time progresses. This work provides a preliminary result of modeling boundedly rational route choice in Nash game, however, the employed road network is too special in the sense that Nash equilibrium is approximately the same as UE given the travel demand (i.e., 100 units) and the rationality parameter α are large enough.

6. Procedural bounded rationality: cognitive search and two-stage cognitive process

Substantive bounded rationality mainly focus on modeling behavioral outcomes. Parameters associated with bounded rationality are assumed to be known. The main goal of the behavioral research is to "estimate parameters we need for fitting theoretical models incorporating known/believed laws to particular situations where we wish to make predictions" (Simon, 1982). Therefore parameter estimation is a critical component of the BR research. Procedural bounded rationality aims to utilize empirical studies to estimate rationality related parameters. A unified framework for route choice decision-making processes adapted from Rasouli and Timmermans (2015) and Zhang (2011) is illustrated in Fig. 4.

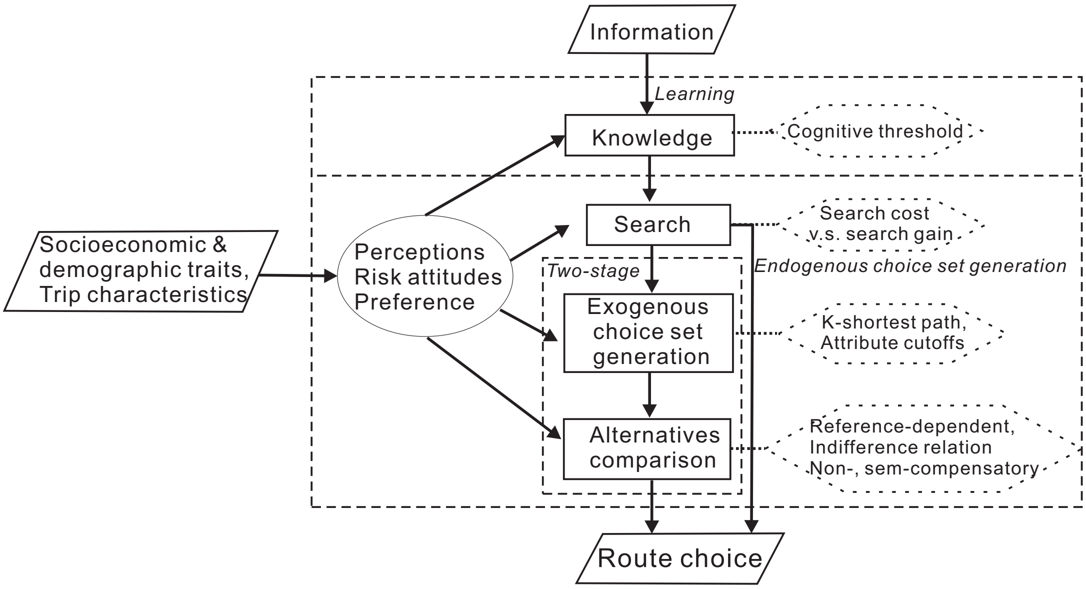


Fig. 4. Boundedly rational cognitive processes.

Before making route choices, travelers are assumed to have some spatial information I via experience, such as network topology, hierarchy of roads, connectivity, alternative paths, and free-flow travel time (i.e., speed limit).

Learning L updates spatial knowledge K on day $t+1$ based on new information obtained on day t: $K^{t+1} = L(I^t, K^t)$.

Built upon knowledge, search S explores alternatives' information and can be defined as a mapping from spatial knowledge K to alternatives: S: $(K^t) \to A$. Define subjective expected search gain B depending on knowledge, i.e., $B = f(K)$, and search cost C. A boundedly rational traveler starts search when expected search gain exceeds search cost.

A route choice process can be regarded as a two-stage process: choice set generation and alternative route choice. It can also be treated as a dynamic search process where travelers keep searching routes till a satisfactory route is found. Accordingly, choice set generation can be categorized into two types: exogenous and endogenous choice set generation. Exogenous choice set generation means choice set is generated before a decision is made. In endogenous choice set generation (Richardson, 1982), decision-making is a dynamic sequential process where a choice set may not be known till the final choice is made.

In the two-stage process, travelers search all feasible paths and form value judgments towards these paths. Define a set of alternative paths as r selected from the choice set $\mathcal{A}$. The choice set can differ among individuals as time goes on, then traveler n's static alternative set is denoted as $\mathcal{A}_n$ and at time instant t as $\mathcal{A}_n^t$. Each alternative r is characterized by an attribute set $\mathcal{K}_r$. The value of attribute k for alternative r is X_{rk}. Due to perception errors, objective and physical attribute values is mapped to cognitive values $g_n(\cdot) : x_{nrk} = g_n(X_{rk}), \forall r \in \mathcal{A}_n, k \in \mathcal{K}_r$. If travelers: (a) have partial knowledge, then $K_{nr} \subset \mathcal{K}_r$; (b) have imperfect knowledge, then $x_{nrk} \neq X_{rk}$; and (c) may not be familiar with all choice alternatives, then $\mathcal{A}_n \subset \mathcal{A}$. Objective or cognitive attribute values are then transformed into a set of value judgments $\{V_{nr}\}$: $V_{nr} = f_n(x_{nrk})$. The utility function can be defined as linear, i.e., $V_{nr} = \sum_k \theta_n X_{rk}$ or nonlinear-in-parameter, i.e., $V_{nr} = \prod_k (X_{rk})^{\theta_n}$. Drivers' risk attitudes, perceptions of alternatives' utilities, and individual preference impact each process and are represented as parameters θ_n, $\forall n$, which are unobservable and enclosed in ovals. Note that if route choice is specified, the notation $\mathcal{P}$ will be used to indicate choice set instead of $\mathcal{A}$.

For ordering choices in order of preference, value judgments are integrated into an overall judgment. It can be a deterministic utility value: $U_{nr} = f_n(V_{nr})$. It can also adapt the deterministic utility value to a random utility: $U_{nr} = V_{nr} + \zeta_{nr}$, where ζ_{nr} is a random error with a specific distribution. Gumbel and normal are two commonly used distributions. Decision rule D determines which alternative to choose based on knowledge: $P(A_*) = D(A, K)$. More specifically, $P(A_*) \triangleq p_{nr} = D_n(U_{nr}, V_{nr}, X_{rk})$, where p_{nr} represents choice probability over the choice set or a boolean value. $D_n(\cdot)$ is the decision-making mapping regarding which choice to take based on all or partial alternatives' utility judgments or even attribute values, resulting in ordering of alternatives.

Bounded rationality thresholds can be embedded into each process, represented by dotted hexagons. We will review how BR is modeled and estimated along the course of cognitive processes. As learning involves time dimension, it will be discussed in Section 7.

6.1. Cognitive search

In the real-world, paths' travel costs are generally stochastic. Some researchers argued that travelers do not take the shortest paths because they are not capable of perceiving actual travel costs due to limited cognitive capacities, or it is too costly to search information over all alternative paths (Gao et al., 2011). A boundedly rational traveler will not start to search path information unless its gain exceeds its cost (Zhang, 2011).

6.1.1. Cognitive cost model

Gao et al. (2011) proposed a cognitive cost model to capture people's route choices in complex contexts with costly information acquisition.

In cognitive cost model, the path travel time is assumed to be random. Before searching, travelers only know the mean and the variance. If they decide to search information, travelers will know the exact travel time of the searched path; however, search consumes cognitive cost. Therefore search action is the trade-off between travel benefit and cognitive cost. Accordingly, the probability of choosing path r given a choice set $\mathcal{P}_n$ for traveler n is mainly composed of two parts:

$$P(r|\mathcal{P}_n) = P(r|\mathcal{P}_n, \text{no search})P(\text{no search}) + P(r|\mathcal{P}_n, \text{search}). \tag{6.1}$$

Assume the ordering of path searching is predefined. At each stage, traveler n has two decide to continue search or stop. At stage 1, no search is performed. If traveler n decides to search information, at stage 2, information on the first ordered path is searched. This process will continue till traveler n chooses to stop searching at some stage t. The maximum number of stages equals to the total number of feasible paths plus one (i.e., no search stage). Each traveler belong to one search class s, which depends on his or her knowledge of the network and flexibility of arrival time. The probability of choosing path r given a choice set $\mathcal{P}_n$ is then further decomposed into the sum of three parts:

$$P(r|\mathcal{P}_n, \text{search}) = \sum_s \sum_{t=1}^{T_n^s} P(r|t, \mathcal{P}_n, H_n^s)P(t|H_n^s)P(H_n^s), \tag{6.2}$$

where,
$P(H_n^s)$: the probability of individual n belonging to search class s;
$P(t|H_n^s)$: the probability of searching path information till stage t given class H_n^s;
$P(r|t, \mathcal{P}_n, H_n^s)$: the probability of choosing path r given information is searched till stage t. Logit models are used to calculate all three probabilities.

In the following, we will mainly introduce the methodologies of computing $P(t|H_n^s)$ and $P(H_n^s)$, because they both involve cognitive process modeling. To reflect that a traveler chooses a satisfying route due to information availability, cognitive constraints and time limit, Gao et al. (2011) adapted a directed cognition model (Gabaix et al., 2006) proposed in economics.

$P(t|H_n^s)$ is computed using a stop-and-go logit model. Let $\mu_{nr}^{t,s}, \sigma_{nr}^{t,s}$ be mean and standard deviation of path travel cost r for individual n from class s at stage t respectively. They are random variables at stage t when search is conducted till $t-1$. At stage t, traveler n decides to continue search or stop. To compute $P(t|H_n^s)$, the utilities associated with these two actions are defined as:

$$\begin{aligned} V(\text{go at } t|H_n^s) &= \theta_{cost}^s + \theta_{benefit}^s B_n^{t,s}(go), \\ V(\text{stop at } t|H_n^s) &= \theta_{benefit}^s B_n^{t,s}(stop), \end{aligned}$$

where,
$B_n^{t,s}(go)$: the expected maximum benefit of searching at stage t, computed in Eq. (6.4a);
$B_n^{t,s}(stop)$: the benefit of stopping search at stage t, computed in Eq. (6.4b);
$\theta_{cost}^s, \theta_{benefit}^s$: cost and benefit coefficients, which need to be estimated.

If search stops at stage t, there is no search cost and thus $V(\text{stop at } t|H_n^s)$ does not contain the term θ_{cost}^s. On the other hand, when search is conducted, a higher benefit will be obtained. $B_n^{t,s}(go)$ and $B_n^{t,s}(stop)$ are defined as follows:

$$B_n^{t,s}(go) = \int_{\mu_{nr}^{t,s}, \sigma_{nr}^{t,s}} \ln \sum_{r\in\mathcal{P}_n} \exp\left[V_{nr}^{t,s}(\mu_{nr}^{t,s}, \sigma_{nr}^{t,s})\right] f(\mu_{nr}^{t,s}, \sigma_{nr}^{t,s}) d\mu_{nr}^{t,s} d\sigma_{nr}^{t,s}, \tag{6.4a}$$

$$B_n^{t,s}(stop) = \ln \sum_{r\in\mathcal{P}_n} \exp\left(V_{nr}^{t-1,s}\right), \tag{6.4b}$$

where,
$V_{nr}^{t,s}(\mu_{nr}^{t,s}, \sigma_{nr}^{t,s})$: the utility of choosing path r for individual n from class s at stage t, which is a linear function of travel time's mean and standard deviation;
$f(\mu_{nr}^{t,s}, \sigma_{nr}^{t,s})$: the joint distribution of path travel time means and standard deviations given the search operation.

$P(H_n^s)$ is also calculated by a logit model. A class membership function is introduced which includes two classes: search or no search. The utilities associated with these two classes are defined as:

$$V_n(\text{search}) = \theta^s_{cost} + \theta^s_{benefit} B_n^{s1}(go) + V(X_n, \theta),$$
$$V_n(\text{no search}) = \theta^s_{benefit} B_n^{s1}(stop),$$

where,
$B_n^{s1}(go)$: expected maximum utility of searching one path;
$B_n^{s1}(stop)$: benefit of no search;
$V(X_n, \boldsymbol{\theta})$: utility related to traveler n's characteristics, such as network familiarity and arrival time flexibility of the trip. $\boldsymbol{\theta}$ is a parameter vector associated with individual characteristics.

Gao et al. (2011) estimated parameters associated with search from simulated revealed preference data among three alternative routes from home to workplace using three models: the cognitive cost model, a no-information model (i.e., travelers do not search information and choose paths based on unconditional travel time distribution), and a fully-information model (i.e., a random utility model where travelers know exactly the realized travel time, however, modelers do not have full information of how they behave). In the cognitive cost model, parameters in class membership function are latent variables and thus some latent model estimation technique was employed. Without latent explanatory variables (i.e., search cost and benefit, network familiarity and schedule flexibility), the no-information model the fully-information model generate estimates with larger variance of error. The higher the search cognitive cost is, the less likely people will search information, and thus the worse the cognitive cost model will perform in prediction. A commuter who is more familiar with the network or has more flexible work schedule tends not to search information. Heterogeneity of search costs was also verified from Twin Cities travel survey (Zhang, 2006). Such finding will provide theoretical basis for deployment of differentiated traveler information provision.

6.1.2. Endogenous choice set search

Richardson (1982) argued that decision-making is a dynamic sequential process where a choice set may not be known till the final choice is made. Accordingly, all searched alternatives is included into a choice set before the search is stopped. This process belongs to “endogenous choice set generation”. Search is involved throughout the process. Define utilities of alternatives as u along with its probability density function as $f(u)$ (e.g., standard normal). Search will not stop till the cost of search, denoted as a constant ε, is beyond expected search gain, denoted as $\bar{B}$. In other words, search will stop if $\bar{B} > \varepsilon$. The expected search gain is computed as:

$$\bar{B} = \bar{u}\sigma = \int_{-\infty}^{\infty} uf(u)du = \int_{u_{\max}}^{\infty} (u - u_{\max})f(u)du, \tag{6.6}$$

where,
$\bar{u}$: the gain of one extra search and is computed as: $\bar{u} \int_{-\infty}^{\infty} uf(u)du = \int_{u_{\max}}^{\infty} (u - u_{\max})f(u)du$;
$u_{\max}$: the maximum utility of the alternatives already searched;
σ: the standard deviation of the actual utility of the alternatives already searched.

Numerical simulations were performed to validate the proposed search process. The similar result was found as in Gao et al. (2011) that higher search cost hinders search more. In addition, the more prior information of alternative utilities a decision-maker has, the more likely he or she will conduct search.

6.2. Two-stage process

According to Ben-Akiva et al. (1984), travelers' route choice behavior is regarded as a two-stage process: path set generation (i.e., a path choice set is generated between origin and destination according to route characteristics) and traffic assignment (i.e., the traffic demands are mapped to these generated paths based on certain traffic assignment criteria). There are some variants of such division. For example, the first stage of path set generation can be further divided into two stages (Hato and Asakura, 2000; Ridwan, 2004): network recognition and alternative generation and elimination. Hiraoka et al. (2002) proposed three stages: alternative formation, judgment process (including extraction of finite environment properties and estimation of criterion value), and decision-making process (i.e., evaluation of alternative route costs or choice of route). In this paper, we will treat route choice as a two-stage process. Its mathematical formulation is (Rasouli and Timmermans, 2015):

$$P(r|\mathcal{P}) = \sum_{\mathcal{P}_n \in 2^{\mathcal{P}_n}} P(r|\mathcal{P}_n)P(\mathcal{P}_n|\mathcal{P}), \tag{6.7}$$

where,
$\mathcal{P}_n$: the consideration path set;
$2^{\mathcal{P}_n}$: the set of all possible path sets;
$P(r|\mathcal{P}_n) \triangleq p_{nr}$: the path choice probability;
$P(\mathcal{P}_n|\mathcal{P})$: the alternative path set choice probability.

6.3. Exogenous choice set generation

In the two-stage theory, choice set generation is generated before a decision is made, i.e., "exogenous choice set generation". In the following, we will introduce models to generate exogenous choice set.

6.3.1. K-shortest path algorithms

The K-shortest path algorithms, widely used in path set generation, search for the first K paths with the least path costs. This algorithm relaxes the requirement of obtaining the shortest paths and reduces computational burdens, which reflects the BR principle.

Using GPS data collected in Minneapolis, MN, Zhu (2011) evaluated four K-shortest path algorithms:

1. link labeling approach (Ben-Akiva et al., 1984): generates K shortest paths by minimizing multiple objectives, including shortest travel time, shortest free-flow travel time, shortest distance, the number of intersections and so on.
2. link elimination (Azevedo et al., 1993): "generates the Kth shortest path by removing all links of the first $K-1$ shortest paths from the network".
3. link penalty (de la Barra et al., 1993): "generates shortest paths after multiplying the link travel time of the current shortest path by 1.05."
4. simulation (Zhu, 2011): each link's travel time is assumed to follow a normal distribution with mean computed from ensemble GPS speed data and variance as 15% of mean. A path travel time is the average of multiple draws from link distributions. The path with the shortest average travel time will be chosen.

A measure, named "overlap thresholds of predicted and actual chosen paths", is used to compare the performance of each method. Simulation is shown to generate the highest overlap threshold, which covers 63% observed paths.

6.3.2. Non-compensatory heuristic

The non-compensatory heuristic is commonly employed in choice set generation to reduce the size of the problem. Its principle is that only a partial set of alternatives are selected based on attribute values. It includes two strategies (Rasouli and Timmermans, 2015): conjunctive (i.e., an alternative is selected only when all attributes meet requirements), i.e., $p_r = \begin{cases} 1, X_{rk} \geqslant \varepsilon_k, \forall k \in \mathcal{K}_r \\ 0, \text{ o.w.} \end{cases}$; and disjunctive (i.e., an alternative is selected if at least one attribute meets requirements), i.e., $p_r = \begin{cases} 1, X_{rk} \geqslant \varepsilon_k, \exists k \in \mathcal{K}_r \\ 0, \text{ o.w.} \end{cases}$. Satisfactory heuristic (Simon, 1955) is one example of conjunctive strategy as an alternative is selected if it meets the minimum aspiration level on all attributes (Simon, 1955).

6.4. Route choice decision strategies

At the stage of route choice, two decision strategies are employed: compensatory, non-compensatory. There exists a third strategy which is the mix of compensatory and non-compensatory strategies: semi-compensatory. The definitions of three strategies are:

1. Compensatory strategy: there exist trade-offs among attributes. In other words, the attribute of one alternative can be compensated by another attribute. The traditional random utility maximization framework utilizes the compensatory strategy;
2. Non-compensatory strategy: each alternative is treated as a set of attributes (i.e, aspects). Alternatives are selected attribute-by-attribute. For one alternative, its superior attribute cannot compensate its inferior attribute. The non-compensatory strategy can be heuristic as only a partial set of alternatives are compared to obtain an optimal one. It can also be exhaustive, for example, satisfactory heuristic (Simon, 1955) selects an alternative that meets the minimum requirements on all attributes;
3. Semi-compensatory strategy: the non-compensatory strategy is employed for choice set generation and the compensatory strategy is used to find an optimal choice.

6.4.1. Compensatory strategy

The discrete choice model is a common tool to model people's choice behavior and estimate behavioral parameters. Within the framework of discrete choice modeling, the random utility maximization model (RUM) is adopted in modeling and predicting drivers' route choice behavior among a set of finite paths. Provided perception errors are Gumbel distribution, RUM can be expressed in the form of a multinomial logit model. The logit model assumes paths are independent of each other and has independence from irrelevant alternatives (IIA) property. In reality, however, many paths overlap with each other and are thus not independent. To overcome this limitation, various RUM models were proposed: C-logit (Cascetta et al., 1996), path-size logit (Ben-Akiva and Ramming, 1998), nested logit (Jha et al., 1998), cross-nested logit (Vovsha and Bekhor, 1998), multinomial probit (Cascetta, 1989; Daganzo et al., 1977; Jotisankasa and Polak, 2006), and mixed logit or multinomial probit with logit kernel (Ben-Akiva and Bolduc, 1996). However, all these models assume decision-makers are fully rational and fully informed, and most importantly, they are utility-maximizers (Swait, 2001).

The generic framework of the discrete choice model for utility maximization is:

$$P(A_r) = P(U_{nr} > U_{nr'}) = P(V_{nr} + \zeta_{nr} > V_{nr'} + \zeta_{nr'}). \tag{6.8}$$

Different specifications of error structure, ζ_{nr}, give different models. Gumbel distribution gives logit model while normal gives probit model. The utility function can be defined as linear, i.e., $V_{nr} = \sum_k \theta_n X_{rk}$ or nonlinear-in-parameter, i.e., $V_{nr} = \prod_k (X_{rk})^{\theta_n}$. Maximum likelihood is used to estimate coefficients defined in the utility V_{nr}.

The discrete choice model may be unrealistic because:

1. the utility associated with a choice is associated with a '*reference point*' (Section: Reference-dependent model);
2. decision-makers are indifferent to small differences in utility values (Sections: Indifference relation and Indifference to small changes in dynamic choices) .

In the following, we will introduce how the discrete choice model can be extended to accommodate more realistic choice behavior. Note that there only exist a small amount of existing studies on extending the discrete choice model in route choice, due to its complex nature. However, there are a large body of literature on other travel choices, such as mode choice. We will introduce generic travel choice models and hope readers can apply them to model boundedly rational route choice behavior in their own research.

Reference-dependent model. Reference-dependent models play an important role in boundedly rational behavioral modeling. People generally evaluate attributes/utilities compared to the best value or some reference point instead of using absolute values. Such models can be applied to both route choice and choice set generation.

The relative evaluation value of alternative r with respect to attribute k is defined as a mapping $f_n(X_{rk})$ over the attribute value X_{rk}. Based on relative evaluation, one or more reference points are used to determine the alternatives in attribute or utility space. Such reference points can be categorized into endogenous or exogenous reference points. An exogenous reference point is a constant that has already been calibrated from experimental data. An endogenous reference point needs to be determined from system values, such as a path with the minimum travel time or the maximum attribute value across all alternatives. Relative *attribute* or *utility* threshold/cutoff or aspiration level, denoted as ε, will be introduced to capture people's indifference to alternatives whose evaluation values are below the thresholds. Travelers may adjust aspiration level based on updated knowledge. If some endogenous reference point, such as the maximum attribute value across all alternatives, is chosen as a reference, the relative attribute is: $f_n(X_{rk}) = \frac{\max_{j\in\mathcal{A}_n} X_{jk} - X_{rk}}{\max_{j\in\mathcal{A}_n} X_{jk}}$.

Accordingly the BR principle is reflected as elimination of alternatives whose attributes or utilities are below certain thresholds. The behavioral mechanism includes maximization of relative advantage, maximization of relative utility, minimization of regret (such as maximin, maximax, minimax regret models). Interested readers can refer to Rasouli and Timmermans (2015) for detailed description of each model. The unified framework of reference-dependent models is:

$$p_{nr} = P(\mathbf{E}_r \geqslant \boldsymbol{\varepsilon}) \in [0, 1], \tag{6.9}$$

where,
$P(\cdot)$: the probability of alternative r being chosen or included in a consideration set;
$\mathbf{E}_r$: a relative evaluation in attribute/cognitive/utility space, can be $f(X_{rk})$ or $g(V_{nr})$;
$\boldsymbol{\varepsilon}$: attribute or *utility* threshold/cutoff or aspiration level.

Thresholds can be deterministic or stochastic. Define $\boldsymbol{\varepsilon} = [\varepsilon_1, \dots, \varepsilon_N]^T$ as a vector of random variables with its joint density function as $f(\varepsilon_1, \dots, \varepsilon_N)$.

$$P(\mathbf{E}_r \geqslant \boldsymbol{\varepsilon}) = \int_{E_{r1}}^{\infty}\int_{E_{r2}}^{\infty}\dots\int_{E_{rK}}^{\infty} f(\varepsilon_1, \dots, \varepsilon_K) d\varepsilon_1 d\varepsilon_2 \dots d\varepsilon_N. \tag{6.10}$$

Thresholds can also be fixed or dynamic. If there does not exist an optimal solution, thresholds can be adjusted dynamically till an optimal solution is attained.

Indifference relation. Discrete choice models assume implicitly the existence of a preferable ordering over alternatives even if the difference in utilities is negligible. However, travelers may be indifferent to small differences in utility values or small changes in utilities when some change happens. The utility function needs to be modified to account for similarity of choice alternatives. Also, "alternative *utility* threshold" will be introduced to capture such indifference.

Psychological experiments (Guilford, 1954) showed that people may be indifferent to two alternatives with similar utilities. Ridwan (2004) defined three fuzzy preference relations for two alternatives: strict preference; indifference; incomparability, and a fuzzy choice function was proposed capturing the fuzziness feature of choices to calculate their rankings. Krishnan (1977) proposed a minimum perceivable difference (MPD) model describing travelers' mode choices among two alternatives with two relations: strict preference and indifference. Denote U_r as the utility of alternative $A_r, r = 1, 2$. It can be expressed as the sum of a deterministic component V_r and a random component ζ_r: $U_r = V_r + \zeta_r$. Denote ε as an indifference threshold. The ordering over two alternatives and its associated probability can be defined as follows:

1. $A_1 \succ A_2$ if $U_1 > U_2 + \varepsilon$, A_1 will be chosen with probability 1. The probability of preferring A_1 is computed as: $p_1 = P(\zeta_1 - \zeta_2 > V_2 - V_1 + \varepsilon)$;

2. $A_2 \succ A_1$ if $U_2 > U_1 + \varepsilon$, A_2 will be chosen with probability 1. The probability of preferring A_2 is computed as: $p_2 = P(\zeta_1 - \zeta_2 < V_2 - V_1 - \varepsilon)$;
3. $A_1 \sim A_2$ if $|U_1 - U_2| \leqslant \varepsilon$, A_1 and A_2 will be chosen with probability π and $1-\pi$ respectively. The probability of being indifferent to A_1 and A_2 is computed as: $p_{12} = P(A_1 \sim A_2) = P(|U_1 - U_2| \leqslant \varepsilon) = P(V_2 - V_1 - \varepsilon \leqslant \zeta_1 - \zeta_2 \leqslant V_2 - V_1 + \varepsilon)$.

Given distributions of ζ_1 and ζ_2, the above preferring probability can be computed. Krishnan (1977) adopted the inverted exponential distribution. The probabilities of choosing each alternative is thus computed as:

$$\begin{cases} P(A_1) = p_1 + \pi p_{12}, \\ P(A_2) = p_2 + (1-\pi) p_{12}. \end{cases} \tag{6.11}$$

The above model was applied to estimate commuters' mode choice, car or train, between Lindenwold and Philadelphia in order to evaluate the Lindenwold High Speed Line. The estimated threshold ε is the by-product of the above program. The threshold is estimated as 23, meaning that only when the utility (a linear function of travel time and travel cost) of the high speed line is 23 higher than that of cars, people will be likely to take the train to work. It predicts better than the logit model in more than 75% of the cases.

The MPD model can only accommodate two alternatives. Lioukas (1984) extended it to the multinomial logit model with more than two choices:

$$P(A_r) = \frac{\exp(V_{nr})}{\sum_{r'} \exp(V_{nr'}) + \sum_{r'} \exp(V_{nr'} + \varepsilon)}, \forall r' \neq \mathcal{P}_n. \tag{6.12}$$

Equations (6.11) and (6.12) describe individuals' static decision-making results. When some change happens to alternatives, such as a new alternative is introduced or the value of one attribute has changed, people may also be indifferent to small changes in utility values.

Indifference to small changes in dynamic choices. Due to existence of inertia, people may place higher weight on the alternative he or she regularly uses, which introduces state dependence and serial correlation in dynamic choices. This dependence can be captured by an inertial variable depending on the utility difference between chosen alternative and others. Cantillo et al. (2007, 2006) further specified the threshold ε in Eq. (6.12) by introducing an inertial variable with state dependence (due to inertias, Cantillo et al., 2006) and serial correlation (due to persistence of unobservable attributes across a sequence of choices, Cantillo et al., 2007).

The inertia variable, denoted as $\varepsilon^{r'r}_{n,t+1}$, is represented by a random utility function depending on the utility difference, i.e.,

$$\varepsilon^{t+1}_{n,r'r} = \varepsilon^{t+1}_{n,r'r}(V(X_n, Z^t_n, \theta^t_n), V^t_{nr} - V^t_{nr'}), \tag{6.13}$$

where,
$V(\cdot)$: a utility function;
X_n: individual characteristics for traveler n;
Z^t_n: trip features for traveler n on day t;
θ^t_n: a vector of parameters needed to be estimated;
V^t_{nr}: the deterministic utility for the alternative r.

Denote the utility of the alternative A_r as $U^t_{nr} = V^t_{nr}(X_n, Z^t_n, \theta^t_n) + \zeta^t_{nr}$, where ζ^t_{nr} is the error term. The traveler n picks a choice r on day t based on a multinomial logit model. A change happens to some attribute attached to an alternative mode on day $t+1$. Assume travelers make stable mode choice right after the change is made. Due to inertia, the probability of switching from the current choice $A^t_n = r$ to $A^{t+1}_n = r', r' \neq r$ is equivalent to:

$$U^{t+1}_{nr'} - U^{t+1}_{nr} \geqslant I^{t+1}_{n,r'r}, \tag{6.14a}$$

$$U^{t+1}_{nr'} - U^{t+1}_{nj} \geqslant I^{t+1}_{n,r'r} - I^{t+1}_{n,jr}, \forall j \in \mathcal{P}^t_n, j \neq r, \tag{6.14b}$$

where,
$I^{t+1}_{n,r'r}$: the inertia variable, represented by a random utility function depending on the utility difference, i.e., $I^{t+1}_{n,r'r} = \gamma_n(V(X_n, Z^t_n, \theta^t_n), V^t_{nr} - V^t_{nr'})$, where γ_n is an unknown coefficient varies randomly among individuals and $V(\cdot)$ is a utility function;
X_n: individual characteristics for traveler n;
Z^t_n: trip features for traveler n on day t;
θ^t_n: a vector of parameter needed to be estimated;
V^t_{nr}: the deterministic utility for the alternative r.

Given the error vector ν_n as Gumbel, on day $t+1$, the conditional probability of choosing the same alternative r as on day t and choosing a different alternative r' are:

$$P(A^{t+1}_n = r|\nu_{nr}) = \frac{\exp(V^{t+1}_{nr} + \nu_{nr})}{\sum\limits_{A_j \in A, j \neq r} \exp(V_{nj} - \varepsilon^{t+1}_{n,jr} + \nu_{nj}) + \exp(V^{t+1}_{nj} + \nu_{nj})}, \tag{6.15a}$$

$$P(A_n^{t+1} = r' | \nu_{nr'}) = \frac{\exp(V_{ir'}^t - \varepsilon_{n,r'r}^{t+1} + \nu_{nr'})}{\sum_{A_j \in A, j \neq r} \exp(V_{nj}^{t+1} - \varepsilon_{n,jr}^{t+1} + \nu_{nj}) + \exp(V_{nj}^{t+1} + \nu_{nj})}, r' \neq r. \tag{6.15b}$$

This model was used to describe travelers' mode switching choice after a new mode is added. Two sets of data were collected in Cagliari, Italy: the RP data in terms of people's mode choices among car, bus, and train; the SP data inquiring the choice between a new train service and the current mode choice. Estimation results showed that a misspecified model without inertia and serial correlation may lead to biases and errors when a newly implemented policy has a substantial impact. The inertia mean $\bar{\gamma}_n$ is 0.67 while inertia standard deviation σ_{γ_n} is 0.81. The standard deviation is higher than the mean. One possible explanation is that a small portion of drivers have inertia but others not. Moreover, "an RP/SP model may underestimate the mean of the inertial effect and overestimate its variance" because SP survey results may have bias which predicts that more people tend to overcome inertia if a substantial change happens in alternatives than it would actually happen.

Di (2014) and Di et al. (2015b) proposed a boundedly rational route switching approach to model commuters' route switching choice after a new route was introduced. This model assumed that commuters would not switch to the new bridge unless the time saving by taking the new bridge was higher than an indifference band. To validate this behavioral assumption, route choices of 78 subjects from a GPS travel behavior study were analyzed before and after the addition of the new I-35W Bridge. Here we will only introduce the work presented in Di et al. (2015b).

Di et al. (2015b) assumed indifference band follows lognormal distribution and presented two ways for parameter specification. The first model assumes everyone's indifference band is drawn from a population indifference band and the estimated population indifference band has mean of 6.56% and standard deviation of 0.17%. The second one specifies that each individual's indifference band depends on their bridge related experiences. Data has shown that in the context of the re-opening of the I-35W Bridge, people's bridge related experiences such as whether they used the old bridge and whether they were afraid of driving on bridges influence their route switching behavior significantly. However, limited by the sample size, the indifference band does not show any dependence on people's demographic information.

Note. According to von Neumann–Morgenstern utility theorem (or the "rational man" paradigm), in the context of choices under uncertainty, the expected utility can be used to define the order of preferences for a rational decision-maker, if four axioms are fulfilled: completeness, transitivity, independence, and continuity. One or more of these axioms may be violated for a boundedly rational decision-maker. For example, under indifference relation or weak preference, alternatives with extreme choice probabilities may not be chosen. In other words, the expected utility cannot represent choice outcomes (Rasouli and Timmermans, 2015). Choosing one alternative from a reduced choice set due to expensive search cost also conflicts with the "rational man" paradigm (Rubinstein, 1998). In reference-dependent models, the dependency of one alternative's preference on its relative position to other alternatives violates these axioms as well (Rasouli and Timmermans, 2015). However, satisficing heuristic with aspiration levels still defines a "rational man" (Rubinstein, 1998). Such a paradigm is commonly discussed in economics literature. However, there exists very little literature in transportation on analyzing whether any bounded rationality models violate the "rational man" paradigm. To the best of our knowledge, we only found one study discussing it, which is Rasouli and Timmermans (2015). It will not be the focus of this paper. However, exploring the relationship between the "rational man" paradigm and a newly developed boundedly rational framework is imperative.

6.4.2. Non-compensatory strategy

Non-compensatory and semi-compensatory strategies discuss order of attribute evaluation and attribute cutoff selection. If there does not exist one optimal solution, decision-makers are assumed to either adapt attribute cutoff or violate cutoff to a certain degree.

Sequential elimination attribute-by-attribute. Decision-making is a dynamic decision process. When attributes are considered sequentially and partially, the BR principle is reflected that only a partial set of attributes is considered if alternatives are selected based on more important attributes. People examine alternatives sequentially by comparing the attributes of one alternative against a set of minimally acceptable standards for each attribute. In lexicographic model Tversky (1972), attributes of interest are specified in order and the alternatives without the selected attributes are eliminated. At each stage, one attribute is specified and the alternatives without the selected attributes are eliminated. Attributes can also be assigned to numerical values as importance levels and are ordered by such subjective rating (Recker and Golob, 1979). At each stage, one attribute of all alternatives is evaluated and alternatives which fail to satisfy the tolerance at a particular importance level are removed from the choice set.

If there exists multiple alternatives after comparing the first attribute, there exist two rules of eliminating alternatives: static or dynamic mechanism. In static mechanism, the attribute with the second importance level is compared, and the third is compared ... till one alternative is eventually picked. It may also end up with no alternative left. Dynamic mechanism allows people to adjust the standards to reality so that a single alternative will be guaranteed (Simon, 1972). In the following, we will briefly introduce how dynamic mechanism works in transportation behavior proposed by Recker and Golob (1979).

Denote the attribute with the ith importance ranking as $k(i)$. At each stage, one attribute of all alternatives is evaluated, alternatives which fail to satisfy some tolerance at a particular importance level are removed from the choice set. Critical tolerances will be adjusted till a single optimal choice is found and thus the decision-making dynamic is iterative.

Denote the attribute with the ith importance ranking as $k(i)$. The key of this model is to estimate the appropriate critical tolerances $\varepsilon_{nk(i)}, k(i) = 1, \dots, K$. Because the output of the model is a single alternative not a probability over a choice set, standard estimation techniques fail. Assume $\varepsilon_{nk(i)} \sim$ some distribution:$(\bar{\varepsilon}_{k(i)}, \sigma_{k(i)})$. Then the problem is equivalent to determine the mean values of these critical tolerances that are generalizable to groups in the population as a whole, i.e., estimates of $\bar{\varepsilon}_{k(i)}, k(i) \in \{1, \dots, K\}$, such that $\sigma_{k(i)}, k(i) \in \{1, \dots, K\}$ of the individual tolerances $\varepsilon_{nk(i)}, k(i) \in \{1, \dots, K\}, n = 1, \dots, N$ are minimized, subject to the constraint that every decision maker is assigned his or her chosen choice.

Two datasets were used to test the proposed sequential decision model: Dataset I was individual's choice of vehicles, Dataset II was travelers' choices of grocery stores from a mail survey. Both datasets included attitudinal data of attributes' ratings. Estimation results had similar pattern that attributes with higher median importance levels exhibit greater mean critical tolerances. Such a conclusion makes sense that people tend to have higher critical tolerances for those attributes they think more important to them. Though the relative comparison of mean critical tolerances is reasonable, no general conclusion can be drawn from the absolute magnitude of these tolerances.

Elimination-by-aspects. In sequential elimination of alternatives by attribute, the final choice highly depends on the ordering of attributes examination. Instead, Tversky (1972) proposed a probabilistic model, i.e., "elimination-by-aspects (EBA)", which assumes no fixed prior ordering of attributes. In EBA model, the probability of examining one attribute is proportional to its importance, therefore more important attributes are more likely of being considered earlier. However, individuals rank the importance of attributes differently, consequently their attribute-ordering procedure is probabilistic and results in different optimal choices. Thus a set of probabilities over alternatives is generated:

$$P(A_r|A) = \frac{\sum_{k \in X' - X^0} X(k) P(k|A_k)}{\sum_{k' \in X' - X^0} X(k')}, \tag{6.16}$$

where,
X': the attributes that belong to at least one alternative in the choice set;
X^0: the attributes that belong to all alternatives in the choice set;
A_k: the alternatives that contain the attribute k.

Young (1984) applied this model to residential location choice and illustrated how this model works in a case with three residential location choices. Assume three alternatives are: $A = \{x, y, z\}$ and each alternative is composed of a collection of attributes: $x = \{\bar{x}, \overline{xy}, \overline{xz}, \overline{xyz}\}, y = \{\bar{y}, \overline{xy}, \overline{yz}, \overline{xyz}\}, z = \{\bar{z}, \overline{xz}, \overline{yz}, \overline{xyz}\}$, where $\bar{x}$ are the attributes that only belong to the alternative x, $\overline{xy}$ are the attributes that only belong to the alternatives x and y, and $\overline{xyz}$ are the attributes that only belong to the alternatives x, y, z. Other notations share the similar meanings. Define $K = \sum_k X(k)$, which is the sum of the importance values over the relevant attributes. As the attributes common to all the alternatives do not affect choice probabilities, attribute $\overline{xyz}$ may be omitted from the summation and the remaining analysis. Then the alternative x can be chosen with three possibilities:

1. Attribute $\bar{x}$ is selected in the first stage, its probability is: $P_1(x) = \frac{X(\bar{x})}{K}$;
2. Attribute $\overline{xy}$ is selected and then x is chosen over y, its probability is: $P_2(x) = \frac{X(\overline{xy})P(x|xy)}{K}$;
3. Attribute $\overline{xz}$ is selected and then x is chosen over z, its probability is: $P_3(x) = \frac{X(\overline{xz})P(x|xz)}{K}$.

$$P(x|xy) = \frac{X(\bar{x}) + X(\overline{xz})}{X(\bar{x}) + X(\overline{xz}) + X(\bar{y}) + X(\overline{yz})}, \tag{6.17a}$$

$$P(x|xz) = \frac{X(\bar{x}) + X(\overline{xy})}{X(\bar{x}) + X(\overline{xy}) + X(\bar{z}) + X(\overline{yz})}. \tag{6.17b}$$

The total probability of selecting x is:

$$P(x|xyz) = \frac{X(\bar{x}) + X(\overline{xy})P(x|xy) + X(\overline{xz})P(x|xz)}{K}. \tag{6.18}$$

The probabilities of choosing y or z can be derived in the same way.

Using the similar concept of critical tolerances in sequential selection model (Recker and Golob, 1979), the probability of choosing alternative x depends on the critical tolerance, i.e., $P(x|xyz) = P(X_{rk} \geqslant (1 - \varepsilon_k) \max_r X_{rk'})$, where X_{rk} is the sanctification level of the kth attribute of alternative r for individual n. The tolerance ε_k is generalized to be a Weibull random variable and its mean and standard deviation can be estimated from a survey of 716 male new residents for residential location choice in Melbourne in 1977–78. The significant attributes included thirteen factors, such as friends, air, trees, shops and so on. These attributes' tolerances vary from 0.2 to 1.0 with a standard deviation of 0.1.

6.4.3. Semi-compensatory strategy

The number of feasible choice set increases exponentially with the number of alternatives and thus choice set selection formation is computationally intensive, especially for spatial choice (e.g., route choice or destination choice). Adding constraints to attributes is one way of reducing the cost of generating choice set.

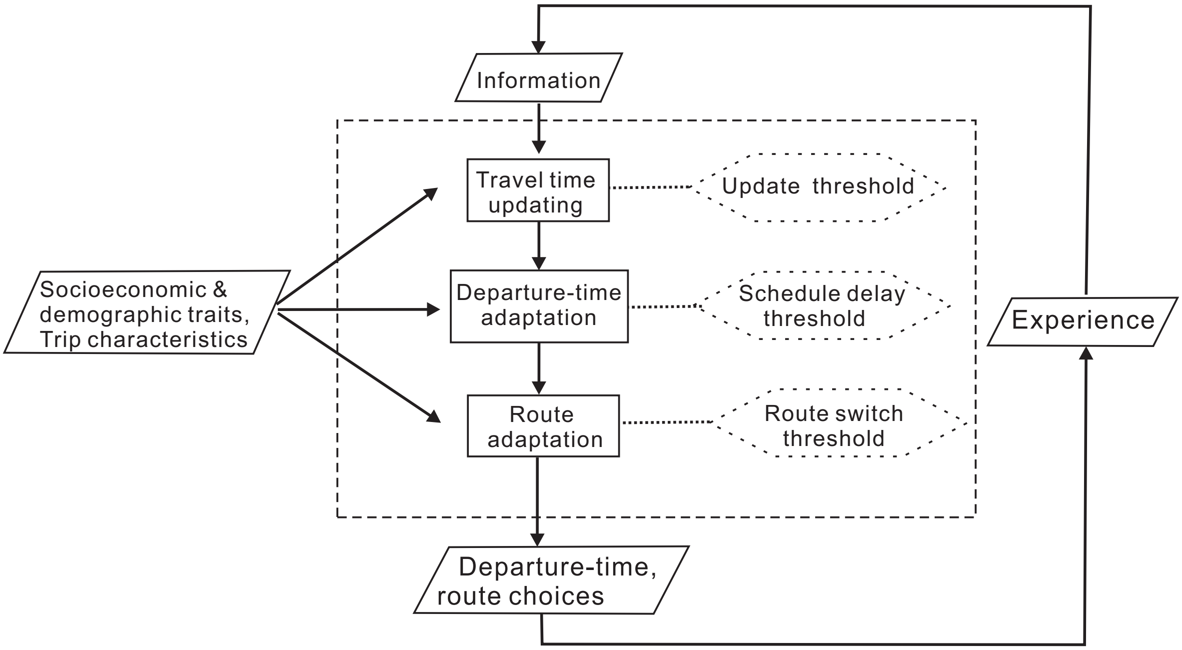


Fig. 5. Boundedly rational travel learning process.

Assume the kth attribute is constrained by upper and lower cutoff points, i.e., $\underline{x}_{rk} \leqslant X_{rk} \leqslant \bar{x}_{rk}, k \in \mathcal{K}_r$. For example, the distance of the alternative path should be within 1.33 times the shortest distance, or the number of left or right turns cannot exceed 5 times (Hato and Asakura, 2000).

The semi-compensatory strategy adopts the non-compensatory strategy for stage one and the compensatory strategy for stage two:

Stage 1 Alternatives are eliminated whose attributes do not satisfy $\underline{x}_{rk} \leqslant X_{rk} \leqslant \bar{x}_{rk}$ and the choice set is reduced using the non-compensatory strategy. For example, conjunctive heuristic can be employed, i.e., only the alternatives whose every attribute meets the constraints will be included in the choice set: $r \in \mathcal{P} \Leftrightarrow \prod_{k\in\mathcal{K}_r} I(\underline{x}_{rk} \leqslant X_{rk} \leqslant \bar{x}_{rk}) = 1, \forall k \in \mathcal{K}_r$.

Stage 2 An optimal choice is made using the compensatory strategy of utility maximization.

In the two-stage optimization, attribute cutoffs must be satisfied and cannot be violated. However, it may not generate an optimal alternative. Then people tend to either adapt thresholds or violate them. In semi-compensatory strategy, thresholds are assumed to be fixed but treated as "soft" constraints. In other words, they can be violated to a certain degree. Accordingly two-stage optimization can be reduced to one stage (Martínez et al., 2009; Swait, 2001) by converting attribute constraints to linear (Swait, 2001) or nonlinear penalties (Martínez et al., 2009) embedded into utility functions. Swait (2001) added a penalized term to the original utility function to accommodate attribute constraints:

$$V_{nr}(X_{rk}) \triangleq V^*_{nr}(X_{rk}) + \sum_{k\in\mathcal{K}} p_{nr}(w_k\lambda_{rk} + \upsilon_k\kappa_{rk}), \tag{6.19}$$

where,

$\lambda_{rk} \geqslant 0, \kappa_{rk} \geqslant 0, \forall r \in \mathcal{A}, k \in \mathcal{K}$: dual variables of two constraints, respectively;

$w_k, \upsilon_k, \forall k \in \mathcal{K}_r$: marginal disutilities of violating the kth lower and upper cutoff constraints, respectively.

The penalized term generates "kinks" in the utility function with respect to parameters. Such non-smooth property can pose difficulty in utility parameter estimation. Instead, Martínez et al. (2009) proposed a nonlinear penalty: $\bar{V}_{nr}(X_{rk}) \triangleq V^*_{nr}(X_{rk}) + \log(\phi_{nr})$, where $\phi_{nr} = \frac{1}{1+\exp(\underline{x}_{rk}-X_{rk})+\rho_k}$ and ρ_k is a sufficiently small non-zero constant. ϕ_{nr} represents the probability of violating cutoffs. This nonlinear utility function is continuously differentiable and facilitates optimization.

A SP survey of 264 customers was conducted for the choice among tree auto rental agencies. The proposed penalty model could capture the heterogeneity effects and was thus able to explain more variability than a random coefficient model.

7. Procedural bounded rationality: learning processes

Making travel decisions is a repeated learning process and three stages are usually considered during dynamic decision-making processes (illustrated in Fig. 5). These stages are indicated in rectangles and enclosed in a big box with dotted borders because they are unobservable. Individual's socioeconomic characteristics, available information and their route choices are indicated by solid parallelograms, representing observable inputs or outputs. Before making decisions, travelers are assumed to have some knowledge of networks from previous experiences. Salient information and new experiences may trigger travelers' update mechanism. Then adaptation to switch departure time or route is made based on certain learning principle, which will provide more information to the next decision-making process. Due to the existence of habit, there exists a threshold at each stage to capture more realistic behavior, indicated in dotted hexagons.

The models discussed in this section will solely adopt reinforcement learning.

7.1. Day-to-day departure-time and route choices

When travel choices are made repeatedly, travelers adjust their behavior accordingly based on previous experiences. Therefore habit plays a critical role and a threshold representing habitual choice behavior should be introduced to characterize habit. In this section, we will discuss how BR is embedded into travel time updating and choice adaptation stages and how the BR parameters are estimated from observed choices by employing stochastic day-to-day dynamics framework.

In the repeated learning process illustrated in Fig. 5, it is found that, as a result of habit, commuters will not update their travel time perception if the perceived one minus the predicted one is within a threshold. Moreover, they will not adjust their departure time unless the difference between the preferred arrival time and the actual arrival time exceeds a bound (Chen and Mahmassani, 2004; Hu and Mahmassani, 1997; Jayakrishnan et al., 1994; Mahmassani and Chang, 1987; Mahmassani and Jayakrishnan, 1991; Mahmassani and Liu, 1999; Srinivasan and Mahmassani, 1999; Yanmaz-Tuzel and Ozbay, 2009). This bound for lateness and earliness are different and people are usually more sensitive to lateness. In addition, commuters will not switch routes if the difference between perceived travel time and experienced one exceeds a bound (Akiva, 1994).

In the following, modeling travel time perception update and departure-time and route choice adaptation will be introduced along with parameter estimation.

7.1.1. Travel time perception updating

Jha et al. (1998) assumed that individuals update their travel time whenever new traffic information is obtained or new travel time is perceived. This assumption is unreasonable due to the existence of updating costs. Chen and Mahmassani (2004) and Jotisankasa and Polak (2006) proposed that only salient information impacts travelers. Assume travelers do not update travel time if the difference between the perceived travel time $\hat{C}_n^t$ and the experienced travel time C_n^t exceeds some threshold. Let $y_n^{t,time}$ denote a travel time update indicator for traveler n at the end of day t, which equals to 1 if traveler n updates travel time after day t, and 0 otherwise:

$$y_n^{t,time} = \begin{cases} 0, \text{ if } \hat{C}_n^t - C_n^t \leqslant \varepsilon_{n,o}^{t,time}, C_n^t - \hat{C}_n^t \leqslant \varepsilon_{n,u}^{t,time}, \\ 1, \text{o.w.} \end{cases} \tag{7.1}$$

where $\varepsilon_{n,o}^{t,time}, \varepsilon_{n,u}^{t,time}$ denote travel time overestimation and underestimation thresholds for user n on day t respectively. Denote $\varepsilon_n^{t,time}$ as a fraction threshold instead of an absolute value, then travel time updating only happens if $|\hat{C}_n^t - C_n^t| \leqslant \varepsilon_n^{t,time}\hat{C}_n^t$. In this case, overestimation and underestimation share the same fractional threshold.

If travel time is random, Chen and Mahmassani (2004) proposed another updating mechanism that a traveler will not update travel time perception until his or her confidence in all path travel times is below a desired level, i.e., $\alpha_{nr}^t \leqslant \frac{1}{\sigma_n \bar{C}_{nr}^t}, r \in \mathcal{P}$, where α_{nr}^t is traveler's confidence in path r's travel time, σ_n is the variance of the perceived travel time over a segment of unit travel time, and $\bar{C}_{nr}^t$ is the mean perceived travel time of path r.

Overestimation and underestimation thresholds are random variables depending on individual characteristics and trip features. Therefore they can be expressed as:

$$\epsilon_{n,o}^{t,time} = V_o^{time}(X_n, Z_n^t, \theta_n^t) + \zeta_{n,o}^{t,time}, \tag{7.2a}$$

$$\epsilon_{n,u}^{t,time} = V_u^{time}(X_n, Z_n^t, \theta_n^t) + \zeta_{n,u}^{t,time}, \tag{7.2b}$$

where,
$V_o^{time}(\cdot), V_u^{time}(\cdot)$: deterministic utility functions for overestimation and underestimation;
X_n: individual characteristics for traveler n;
Z_n^t: trip features for traveler n on day t;
θ_n^t: parameters for traveler n on day t;
ζ_n^t: error terms for traveler n on day t, is either $\zeta_{n,o}^t$ or $\zeta_{n,u}^t$.

The probability of updating can be calculated by a multinomial logit or a multinomial probit model. If the traveler decides to update travel time, there are three classes of models in travel time updating: weighted average (Nakayama et al., 2001), adaptive expectation (Nakayama et al., 2001), or Bayesian (Jha et al., 1998; Jotisankasa and Polak, 2006).

If drivers are myopic and only use yesterday's travel cost, updating is reduced to $\hat{C}_n^t = C_n^{t-1}$, i.e., the perceived travel time on day t is equal to the experienced travel time on day $t-1$.

7.1.2. Departure-time and route choice adaptation

After travel time is updated, drivers adjust their departure-time and route choices based on certain rules. There are two classes of research on modeling choice adaptation: the first one is based on utility maximization (Chen and Mahmassani, 2004; Jotisankasa and Polak, 2006) and the second one is based on bounded rationality. We will only focus on boundedly rational departure-time and route choice adaptation.

To study the impact of advanced travel information on people's behavior, Hu and Mahmassani (1997); Jayakrishnan et al. (1994); Mahmassani and Chang (1987); Mahmassani and Jayakrishnan (1991); Mahmassani and Liu (1999) conducted a series of experiments and showed that people are boundedly rational when choosing routes repeatedly with information. These experiments were run on an interactive simulator-DYNASMART, incorporating pre-trip departure time, route choices and en-route path switching decisions. Subjects, as travelers, picked departure time pre-trip based on the previous days' travel experiences and chose paths en-route at each node based on available information.

Let $y_n^{t,dep}$ denote a departure time switching indicator for traveler n on day t, which equals to 1 if traveler n switches departure time on day $t+1$, and 0 otherwise. Assume traveler n will not adjust his or her departure time unless the schedule delay (i.e., preferred arrival time minus actual arrival time) exceeds some threshold. Early and late arrivals have distinct indifference bands, denoted as $\varepsilon_{n,e}^{t,dep}, \varepsilon_{n,l}^{t,dep}$ respectively, representing tolerable schedule delay. Then,

$$y_n^{t,dep} = \begin{cases} 0, \text{ if } T_n^{*t} - T_n^t \leqslant \varepsilon_{n,e}^{t,dep}, T_n^t - T_n^{*t} \leqslant \varepsilon_{n,l}^{t,dep} \\ 1, \text{o.w.}. \end{cases} \tag{7.3}$$

where T_n^{*t}, T_n^t denote preferred arrival time and actual arrival time for traveler n on day t respectively.

Let $y_{nj}^{t,route}$ denote a route switching indicator for traveler n at the intermediate junction node j on day t, which equals to 1 if traveler n switches his or her initial route or route en-route at node j after day t, and 0 otherwise. Assume travelers do not change pre-trip route or path en-route unless the trip time saving (the difference between predicted travel time of the current path and that of the best path from this node to destination) remains within his or her route indifference band:

$$y_{nj}^{t,route} = \begin{cases} 0, \text{ if } C_{nj}^t - C_{nj}^{t,b} \leqslant \varepsilon_{nj}^{t,route} C_{nj}^t, \\ 1, \text{o.w.} \end{cases} \tag{7.4}$$

where,

$C_{nj}^t, C_{njt}^{t,b}$: the trip times of the chosen and the best path for traveler n from node j to destination on day t respectively;

$\varepsilon_{nj}^{t,route}$: the relative indifference band defined as a fraction of C_{nj}^t.

The aforementioned indifference bands vary among the population over time and thus are random:

$$\epsilon_{n,e}^{t,dep} = V_e^{dep}(X_n, Z_n^t, \theta_n^t) + \zeta_{n,e}^{t,dep}, \tag{7.5a}$$

$$\epsilon_{n,l}^{t,dep} = V_l^{dep}(X_n, Z_n^t, \theta_n^t) + \zeta_{n,l}^{t,dep}, \tag{7.5b}$$

where subscripts e, l represent early and late sides. $V_e^{dep}(X_n, Z_n^t, \theta_n^t), V_l^{dep}(X_n, Z_n^t, \theta_n^t)$ are linear-in-parameter functions of user's initial absolute or relative indifference bands, real-time information reliability (defined as the difference between actual travel time and reported travel time from the real-time information system), the travel time difference between today's and yesterday's (myopic factor), age, and gender.

7.1.3. Estimation of departure-time and route choices

There are two schemes of estimating departure time and route choices under real-time information: joint estimation based on joint probability and hierarchical estimation based on conditional probability.

Given specifications for utility errors of switching departure-time and routes, repeated observations of departure time and route switching decisions can be modeled as a multinomial logit or a multinomial probit function. The maximum likelihood estimation is adopted to estimate parameters in utility functions. In Jha et al. (1998), the initial indifference bands for route switching were shown to be one dominant covariate: the relative indifference band for route switching was 19% for the pre-trip route decision and 18% for the en-route decision; the absolute indifference band for departure-time switching is 1 min. If a commuter experienced an increase in travel time in the previous day, he or she was less likely to switch. In addition, male commuters tended to switch routes more frequently than females.

Departure-time and route choices can be also estimated by a hierarchical model (Jha et al., 1998): the probability of selecting departure interval i and path r on day t by individual n is based on a conditional probability instead of a joint probability: $P_{n,ri}^t = P_{ni}^t \times P_{n,r|i}^t$, where, P_{ni}^t is the probability of selecting departure interval i and $P_{n,r|i}^t$ is the conditional probability of selecting route r given the traveler n has selected departure interval i. These two probabilities can be calculated by logit models, respectively.

7.2. Agent-based models

As the cognitive process is complex and involves numerous decision-making rules, agent-based simulation is a useful tool to emulate such behavior and generate equilibrium solutions aside from analytical approaches. An agent-based model includes three elements: agents (i.e., independent or interactive decision-makers), an environment, and rules. Travelers search routes based on a set of rules. Therefore rule-based learning simulation is a natural representation of how each agent makes decisions.

7.2.1. Rule-based learning simulation

Travelers search routes dynamically while adjusting their indifference bands so that an optimal route can be found. To accommodate such a cognitive process, Zhang (2006) proposed a positive search, information, learning, and knowledge (SILK) theory wherein travelers update their spatial knowledge K^{t+1} about network by a Bayesian learning process based on some prior knowledge K^t when new information I is available: $P(K^{t+1}|I) = \frac{P(I|K^t)P(K^t)}{P(I)}$. The search cost is calibrated from a route search process survey conducted in the Twin Cities area. The behavior of searching for and changing routes are defined by 'if-then' rules, calibrated from GPS-based route choice field experiments. One application of the SILK theory is to compute the behavioral user equilibrium (BUE) which is achieved "when all users with imperfect network knowledge stop searching for alternative routes" due to higher perceived search cost compared to the expected gain from an extra search (Zhang, 2011). The simulation was performed in the Twin Cities road network with 7976 nodes, 20, 194 links, and 600, 000 travelers during the peak hour. Simulation results demonstrated that at BUE, the actual travel cost a drivers experiences is on average 18% higher than the minimum path cost, which can go as high as up to 42% in the worst scenario. Only 25% travelers use the minimum travel cost paths.

Zhang (2011) also compared UE, SUE, and BUE. In the Twin Cities road network, UE underestimates congestion level on highly congested links such as freeway bottlenecks. On the other hand, under SUE and BUE, travelers are less sensitive to increased congestion level either due to perception error or search cost.

7.2.2. Stochastic learning automaton

When travel time is assumed to be random, Yanmaz-Tuzel and Ozbay (2009) applied another learning mechanism, stochastic learning automata (SLA), to study drivers' departure-time choice adaptation in response to a toll change on New Jersey Turnpike (NJTPK). Travelers have three options to depart for work: pre-peak, peak and post-peak. Each driver's departure time choice is assumed to be automated by a stochastic learning automaton which generates a sequence of actions based on drivers' past experiences and interactions with the environment.

The transportation system is defined as a random medium where a traveler can decide and update their departure-time. Denote A_n^t as traveler n's departure-time choice on day t. $y_n^{t,dep}$ is the utility experienced from a departure-time choice, which is calculated from the automaton based on personal experiences. The utility is a binary variable in the form of reward ($y_n^{t,dep} = 0$) or punishment ($y_n^{t,dep} = 1$). Drivers will not update their departure time if deviation of actual arrival time from desired arrival time given the selected departure time falls within an indifference band. The probability of an alternative A_n^t being unfavorable is then computed as $P(y_n^t = 1|A_n^t)$, which is estimated by a Bayesian random coefficient model via individual travel surveys in terms of their departure time choices and socio-economic characteristics. Individual characteristics include the amount of toll charged, work schedule flexibility, education, age, employment, and gender. Then the transition probability of departure-time choice $P(A_n^{t+1})$ on day $t+1$ is updated based on a linear reward-penalty reinforcement learning scheme.

Learning parameters introduced in the reinforcement scheme are estimated from drivers' departure time choices observed from NJTPK toll data. This model successfully mimics NJTPK users' day-to-day travel behavior. However, the learning parameters are close to zero, which were quite different from the values in other fields because of biased samples. Most commuter samples this study included were regular commuters, thus they were familiar with traffic conditions and tended to adapt their choice behavior quickly.

8. Discussions

8.1. Summary of methodologies

The existing methodologies reviewed can be categorized into four types: game-theoretical approach (BRUE, IUE, SUE), statistical congestion game (QRE, BRNE), random-utility models (representing compensatory strategy), and non- or semi-compensatory models. All these approaches include five elements and differ in one or combinations of them (Tables 2).

In modeling route choice behavior, which model is adopted depends on the goal. Game-theoretical models and statistical congestion game models focus on outcomes and consider the congestion effect caused by other users. Random-utility models and non- or semi-compensatory models assume individuals independently choose routes but it may depend on trip features or individuals' demographic characteristics. Behavioral related parameters can also be estimated. Non- or semi-compensatory models with bounded rationality simplify the cognitive process of decision-making and have great potential in reducing computational complexity of modeling boundedly rational route choice behavior. Among them, statistical congestion game models are mixtures of game-theoretical models and random-utility models, which capture both individual interaction and statistical aspects of route choice behavior. However, they suffer from critiques. For example, in logit quantal response equilibrium (LQRE), the source of variations in the rationality parameter is unclear: it may come from individuals or from different contexts (e.g., with or without real-time information). Its value may not increase with repeated experiments due to inertia. Therefore quantifying this extra parameter requires deep understanding of rationale underlying choice

Table 2
Methodologies of modeling boundedly rational route choice.

Elements	Game-theoretical models	Statistical congestion game	Random-utility models (i.e., compensatory models)	Non- or semi-compensatory models
Attribute	Travel time	Travel time	Multiple attributes	Multiple attributes
Attribute value	Flow-dependent	Flow-dependent	Flow-independent	Flow-independent
Alternative	All paths	All paths	All or partial paths	All or partial paths
Utility	Accurate	Erroneous or cognitive	Erroneous or cognitive	Accurate
Choice	Deterministic aggregate flow	Choice probability and integer flows	Choice probability	Boolean or choice probability
Category	Aggregate travel demand model	Disaggregate statistical model	Disaggregate statistical model	Disaggregate statistical model
Pros	Captures congestion level	Captures both congestion level and error/stochasticity	Captures error/stochasticity	Simplifies search and reduces choice set
Cons	No demographic information, no parameter estimation	Complex equilibrium calculation and parameter estimation	No individual interaction	Difficult to model heuristics

Table 3
Comparison of bounded rationality related equilibria.

Equilibrium	Application	Issues	Existence or uniqueness conditions	Uniqueness
BRUE	Capture irreversible network change	Solving it is challenging and may be mathematically intractable	Increasing link cost functions	Generally non-unique
BR-DUE	Ensure existence of a DUE	Solving it is challenging and may be mathematically intractable	Increasing link delay functions	Generally non-unique
IUE	Manifest people's inertia in response to information	Individual's inertial path patterns need to be known, which involves cumbersome enumeration of all inertial path patterns and requires nontrivial estimation procedures of such patterns	(Strictly) monotone link cost functions	(Non-)unique
SUE	Commonly used to describe an equilibrium flow pattern which deviates from UE	Not statistical and thus not for parameter estimation	Monotonically increasing cost functions	Unique
QRE/BRNE	Can be used for parameter estimation	Variation sources of rationality parameters are unclear	Monotonically increasing functions	Unique
BUE	Captures searching behavior using positive behavioral model	Not guarantee an equilibrium exists	NA	NA

outcomes. It also lead to a more complex estimation process, as the computation of equilibrium is needed when a likelihood function is constructed. It thus requires a fixed-point algorithm embedded into a maximum likelihood estimation procedure (Seim, 2006).

8.2. Which boundedly rational equilibrium?

Section 4.3 compared the static equilibria using a numerical example. We want to extend the comparison of all equilibria which have been discussed so far to a more abstract level. Each boundedly rational equilibrium prescribes different aspects of bounded rationality, which are illustrated in Table 1. Which equilibrium concept to use depends on which aspect of bounded rationality needs to be highlighted and is illustrated in Table 3.

Remark. SUE is commonly used to describe an equilibrium flow pattern which deviates from UE. However, it is insufficient in some scenarios:

1. It is unique if the path cost is monotonically increasing (Sheffi, 1984). Such uniqueness cannot describe the substantial change in traffic flow in a disrupted or a restored road network (Guo and Liu, 2011).
2. As shown in the relationship between SUE and LQRE, SUE only captures the expected traffic flow without stochasticity and thus it does not contain statistical features. "Stochastic" is actually misleading. It cannot be used for parameter estimation in static traffic assignment. Instead, it is mainly used for traffic flow prediction. On the other hand, SUE is more widely used in stochastic day-to-day dynamics for learning parameter estimation as shown in Section 5.1.2.

8.3. Perfect or bounded rationality?

As numerous evidence shown in Section 2, the failure of explaining realistic travel behavior using perfect rationality motivates bounded rationality in a variety of contexts. However, we need to first answer the philosophical question raised by Simon (1987) before employing any models: *"where the bounds of human rationality are located?"* In other words, *under what conditions shall we need bounded rationality?*

To answer this question, we need to discuss four aspects associated with behavioral models: predictability, transferability, tractability, and scalability.

8.3.1. Predictability

If we need to obtain stable link traffic flows in a fixed network, user equilibrium (UE) should be sufficient. When the network is subject to a significant change due to new road additions, new travel mode construction, or unexpected network disruptions, equilibrium patterns can change substantially and using UE may result in huge deviation from the real equilibrium patterns. Consequently it will result in wrong policy implementation and waste of infrastructure investment. Danczyk et al. (2015) compared drivers' route choice responses in face of non-expected and expected network disruption respectively using real-world data, which was the collapse of the I-35W Bridge and a planned closure of Trunk Highway 36 both in Minneapolis in 2007. It was found that drivers imposed excess travel costs to the roads near the disruption area so that avoidance behavior occurs when the network was disrupted unexpectedly, which was not observed in the planned disruption case. The similar pattern was observed when the new I-35W Bridge was reopened a year later: commuters still tried to avoid the new bridge. In the restored network, UE was not able to predict realistic traffic flows, which motivated the usage of boundedly rational user equilibrium (BRUE) (Di et al., 2015b; Guo and Liu, 2011).

8.3.2. Transferability

Due to inclusion of extra parameters, BR models are quite sensitive to relevant parameters. The misspecified models can result in even worse prediction than PR models. Thus the calibration of indifference bands or cognitive processes is critical in determining the prediction accuracy of models. They can be individual-specific or the same across the entire population. Individual specific parameters require more data and more complicated models to estimate. By far there do not exist sufficient empirical studies on estimation processes due to lack of large amounts of individual route choice data. Therefore we should be cautious when using BR models for the policy-making purpose.

Even a well-specified BR model is calibrated by data collected from one area, a more critical question is, *whether such a model can easily be transferred to another area, context, or time period.* So far there exists only one study which touched upon the issue of transferability from a laboratory experiment to the real-world scenario in route choice study. By comparing commuter departure time and route choice switch behavior in laboratory experiments with field surveys in Dallas and Austin, Texas, Mahmassani and Jou (2000) showed that boundedly rational route choice modeling observed from experiments provided a valid description of actual commuter daily behavior. Such a claim is quite conservative and whether laboratory experimental experiences can truly represent actual commuter daily behavior still remains unclear. In game-playing experiments, McKelvey and Palfrey (1995) found out that the rationality parameter defined in QRE may vary from experiment to experiment. In some games, the rationality parameter grows as time goes on; however in other games, it does not grow as expected. Therefore this parameter can be individual-specific or context-dependent and may not be able to transfer to a different population or a new context.

8.3.3. Tractability

Tractability is one important measure in picking a model. The tractability of BR models is like a two-sided sword. Whether a BR model is mathematically tractable depends on the size of the problem, the emergency of coming up with a solution for policy-making, and most importantly, the behavioral aspects it incorporates. Decision-makers should evaluate the trade-of between accuracy and efficiency.

In substantive rationality, BR equilibria are less efficient to solve than PR equilibria due to existence of indifference bands. Equilibrium is commonly used for policy-making in long-term transportation planning, as it predicts traffic flows and congestion levels within a network, after a change is made (i.e., a toll charge, a lane expansion, a new road addition, or a new travel mode construction). User equilibrium is the most widely adopted because its formulation is a convex program and a variety of algorithms, such as the Frank-wolf algorithm, exist to solve UE efficiently. By introducing the indifference band, equilibria formulation is more complicated because of, for example, non-uniqueness of bounded rationality user equilibrium (BRUE) or numerous potential inertial path patterns each traveler may have for inertial user equilibrium (IUE). Most of the existing studies do not discuss or consider computation complexity of relevant equilibria. Such ignorance hinders the popularity of bounded rationality related equilibria in spite of its prediction power.

We are glad to see that some traffic software starts to take effort in incorporating BR into traffic simulation, such as DynusT (Chiu et al., 2011) and POLARIS (short for "Planning and Operations Language for Agent-based Regional Integrated Simulation"). DynusT was introduced in Section 5.2 and will be skipped here. POLARIS is a transportation system modeling suite developed at Argonne National Laboratory, which is based on an agent-based activity-based travel demand model (Auld et al., 2013). The boundedly rational en-route switching rule using real-time traffic information is employed to model switching behavior. All these efforts make computation of equilibria much easier and should advance BR related research.

In procedure rationality, the cognitive process with the BR principles may be simpler than that with the PR principle, as fewer attributes and alternatives are considered. On the other hand, inclusion of the cognitive process may also complicate modeling and parameter estimation.

8.3.4. Scalability

Scalability of a BR related route choice model deals with the issue such as: *if the total travel demand among one origin-destination (OD) pair is shrunk, will the same value of BR parameters still predict travel behavior correctly?* To answer this question, the scale of BR parameters, more specifically, indifference bands, plays an important role.

In the existing literature, both absolute (Di et al., 2013; Han et al., 2015; Lou et al., 2010) (mainly for equilibrium computation) and relative (Di et al., 2015b; Guo and Liu, 2011; Mahmassani and Liu, 1999) (mainly for parameter estimation) indifference bands are adopted. Whichever scale is used really depends on researchers' purpose.

An absolute indifference band of 1 min means differently in different networks. For a network with an average travel time of 10 min versus one with an average travel time of 100 min, the BR model may be unnecessary for the latter case as 1 min may be in-salient compared to 100 min. Therefore the magnitude of an absolute indifference band may not be very meaningful. In the same network, the average travel time can vary significantly among different OD pairs, thus the relative indifference band is more commonly used. In the existing literature, it ranges from 2.3% (Di et al., 2015b) to 18% (Mahmassani and Liu, 1999). It was found in both Cantillo et al. (2007) and Di et al. (2015b) that the variance of the indifference band may be larger than its mean due to population of heterogeneity.

We need to note that the aforementioned four aspects of evaluating a model are not independent with each other, instead they are correlated. For example, predictability is determined by transferability, tractability, and scalability. Tractability can impact transferability. Scalability is one aspect of transferability. In conclusion, decision-makers should weigh these aspects before choosing a model.

8.4. Substantive or procedural?

Another question of interest raised by Simon (1987) is: "which kind of theory, substantive or procedural, can better predict and explain what decisions are actually reached"? "Are we interested only in the decisions that are reached, or is the human decision-making process itself one of the objects of our scientific curiosity"?

Substantive bounded rationality describes how people *ought to* behave boundedly rational while procedure bounded rationality describes how people *actually* behave boundedly rational from empirical data. Using substantive or procedural bounded rationality depends on the goal. If it is for policy-making in long-term transportation planning, such as a toll charge, a lane expansion, a new road addition, or a new travel mode construction, substantive rationality models are sufficient. If the cognitive process in route choice is the interest, such as path information search or learning, procedural rationality models are desirable. As the cognitive process is usually hidden and cannot be observed directly, some latent variable models need to be employed. Moreover, procedural rationality are empirical studies oriented. Therefore if field or laboratory experimental data is available and parameters are needed to estimate, procedural models should be adopted.

8.5. Behavioral homogeneity or heterogeneity?

Numerous studies reviewed in this paper revealed that people' socio-demographic characteristics influence decision-making, such as age, gender, and network familiarity. Failure to capture such heterogeneity may result in worse prediction. There exist two approaches to accommodate people's taste variations.

1. Random effect models: BR related parameters are assumed to be continuous random variables. Distribution related quantities, such as mean and variance, need to be estimated. To avoid misspecification, longitudinal data for individuals (i.e., repeated route choice observations) and more complicated estimation procedures, such as Markov chain Monte Carlo simulation, are needed.
2. Latent class models: the entire population is divided into a finite number of homogeneous subgroups (i.e., certain latent membership classes) defined by certain ranges of socio-demographic characteristics. Within each class, individuals are assumed to exhibit similar route choice behavior. However, using this type of models for estimation suffers from three issues: identifiability, determination of the number of latent classes, and stationarity (Dillon et al., 1994). Identifiability relates to attainment of unique solutions, including two main aspects: (a) whether a parameterized distribution is identifiable; and (b) whether the estimation procedure can yield unique parameter values. How many latent classes used for estimation is also a critical question. It needs trade-off between accuracy and efficiency. A rough classification of subgroups can generate large within-group heterogeneity. Stationarity assumes travelers have fixed membership classes over time. However, in reality, it may evolve when travelers become more familiar with roads or when real-time information is provided. In this case, non-stationary latent class models are needed.

9. Conclusions and future research directions

Recently there exists a small but growing body of literature on boundedly rational route choice model. However, a unified framework has been lacking. In this paper, we aim to develop a unified framework of boundededly rational route choices based on a comprehensive review of the state-of-the-art of boundedly rational travel behavior modeling.

First, empirical evidence and their references from psychology, economics, and transportation were cited to show that perfectly rational choice behavior cannot reflect realistic aspects of human choice behavior, because of cognitive limits, expensive deliberation costs, non-attainability of perfect rationality via repeated learning process, and infeasible perfectly rational models. On the other hand, people are boundedly rational because of habit, myopia, less cognitive cost, and solution existence.

Then the paper gives an in-depth discussion of all boundedly rational route choice related models in both substantive and procedural rationality. In each rationality category, the static choice and the time-dependent choice are discussed.

Substantive bounded rationality models include static and dynamic traffic assignment. In static traffic assignment, equilibrium is the most concerned concepts. Five equilibria are introduced and compared in a numerical example, each of which represents one aspect of bounded rationality. In dynamic choice, day-to-day/within-day traffic assignment and dynamic congestion games are introduced.

In procedural bounded rationality models, cognitive processes associated with route choice decision-making are modeled, including a two-stage model and learning processes. Three analytical tools are utilized: random utility models based on the compensatory strategy, non-compensatory models, and semi-compensatory models. In addition, agent-based simulation is commonly used in modeling learning behavior.

Based on the reviewed literature, we would like to point out some research gaps which need to be filled in the existing literature.

9.1. Present research gaps

9.1.1. Empirical verification and estimation of bounded rationality

Though there exist several studies which utilize travel survey data to estimate bounded rationality parameters, they are mainly restricted to laboratory data. Nowadays, not only aggregated detector data at fixed locations, but also mobile sensor data from GPS or smart-phones for individual travelers are available. With travel behavioral data from various sources in place, empirical verification of bounded rationality should continue and bounded rationality parameters need to be estimated for various scales of regions.

9.1.2. Boundedly rational route choice behavior modeling under uncertainty

In substantive bounded rationality models, existing studies on analytical properties of BRUE assume that deterministic flow-dependent travel time is the only factor influencing route choices. Two other major contributing factors, travel time reliability and monetary cost, are completely dismissed. These two factors have been incorporated into perfect rationality models and accordingly UE is subjected to many variants: Probabilistic UE (PUE) (Lo et al., 2006), Late arrival penalized UE (LAPUE) (Watling, 2006), Mean-excess traffic equilibrium (METE) (Chen and Zhou, 2010; Chen et al., 2011), Stochastic bicriterion user-optimal (Dial, 1996; 1997) and Bi-objective UE (BUE) (Wang et al., 2009). Significant contributions can be made if these two factors are also incorporated into BRUE.

In procedural bounded rationality models, the expected utility model does not generally consider decision-makers' risk-taking preference, nor does it consider decision-makers' responses to outcome's probabilities associated with their choices. To further capture decision-makers' risk-taking preference, behavioral modeling under uncertainty is desirable. Embedding bounded rationality into choice models under uncertainty to reflect people's risk-taking and different responses should be further explored in route choice study.

Prospect theory introduces a reference point to capture people's loss-gain asymmetry relationship and adds weighting functions to each alternative's utility to describe the fact that people underweights outcomes with a low probability compared to those with a greater probability. It is also introduced to model transport choice, including route choice behavior. Interested readers can refer to Avineri and Ben-Elia (2015) for a comprehensive review on applying prospect theory to route choice behavior. Employing prospect theory or cumulative prospect theory to boundedly rational route choice behavior modeling and traffic assignment equilibrium is one prominent direction.

9.1.3. Transportation network design under bounded rationality

New methodologies are needed regarding transportation network design problem (NDP) with boundedly rational travel behavior. The classical network design problem is usually formulated as a bi-level program: the upper level is the decision made to either enhance capacities of the established links, apply congestion pricing, or add new links to an existing road network; the lower level is an equilibrium problem, describing how travelers are distributed within the new road network. Due to the existence of the indifference band, travelers may respond differently to a network design proposal, leading to non-uniqueness of the equilibrium and causing difficulties in BRUE link flow pattern prediction and proposal evaluation. In existing literature, Lou et al. (2010) was the first to propose a risk-averse congestion pricing scheme in bounded rationality, however, the mathematical properties of BR-NDP have not been fully explored. To better implement BR-NDP, the mathematical properties of BRUE set are needed. Di et al. (2013) showed that the BRUE set can be decomposed into multiple convex subsets provided with affine link cost functions and Di et al. (2016) utilized this property to solve toll pricing assuming risk-averse and risk-prone attitudes. However, such analysis heavily relies on the topological properties of the BRUE set and it can become quite complicated when general link cost functions are employed.

With the proliferation of rule-based agent-based simulation that incorporates bounded rationality and simulation-based optimization, BR-NDP may be solved more easily in complex large-scale networks. Though simulation is able to emulate complex boundedly rational non-stationary behavior, they cannot explain the rationale underlying the simulated phenomena and thus cannot replace analytical methods.

The above three directions are mainly focused on generalizing topics mentioned in this paper. Boundedly rational travel behavior is still understudied and broader research directions need to be provided.

9.2. Future research directions

In addition to the research gaps in existing literature, there also exist many promising research directions on bounded rationality which are worth of exploration in future.

9.2.1. Cognitive process modeling

Bounded rationality, involving extensive psychology and behavorial aspects, has been well-studied in economics and psychology for decades. However, the cognitive process of boundedly rational travel behavior remains understudied in transportation. In other words, more studies on procedural bounded rationality should be developed because the emerging technologies are transforming people's route choice behavior in many ways:

- With the popularity of smartphones and other social media tools (i.e., Waze), real-time information provision becomes more common, which facilitates information search process modeling. Travelers' compliance to en-route information also involves cognitive processes;
- Provided with rich traffic information, decision-makers spend more time in generating alternatives than in making final decisions (Simon, 1987);
- Information provision can alter travelers' day-to-day or within-day travel decisions, which requires dynamic learning models.

Moreover, non- and semi-compensatory strategies represent more nature psychological aspects of decision-making and are crucial in simplifying travelers' route choice processes. However, they are not fully explored in modeling route choice. Most of existing literature focused on travel choices other than route choice, such as residential location choice (Young, 1984) or destination choice (Recker and Golob, 1979). The intricacies of incorporating non-compensatory or semi-compensatory strategies into complex route choice processes should be explored.

In recent years, there are several studies which aim to model route choice cognitive processes in very different schemes compared to the methods introduced in this paper. For example, Manley et al. (2015) assumed that travelers obtain route information based on some hierarchical model of urban space. At the strategic level of route planning, a traveler first selects regions where a destination resides. Then influential nodes, including gateways (i.e., major roads connecting regions), are selected. The complete route to the destination is the one with the shortest distance connecting all influential nodes. Heuristics, such as elimination by aspects or non-compensatory strategies, are employed at each level of decision. In addition, analyzing verbal reports collected from ten subjects in an laboratory experiment with semantic content analysis, Senk (2010) pointed out that route choice is process-oriented wherein a list of strategies are adopted by travelers. These efforts shed light on marriage of psychological research and transportation engineering research.

9.2.2. Boundedly rational stochastic game-theoretical modeling

The boudedly rational game-theoretical model results in boundedly rational Nash equilibrium and its analytical properties are generally tractable. With these elegant properties, Zhao and Huang (2014) opened up a new direction for borrowing the game-theoretical framework to model boundedly rational route choice behavior. But it is simply a direct application of quantal response equilibrium (QRE) whereas travelers are finite instead of infinitesimals assumed in stochastic user equilibrium (SUE). To establish the relationship between QRE and SUE, large population approximation may be needed.

QRE is a statistical version of game-theoretical equilibrium. Such integration facilitates parameter estimation in interactive game. Therefore QRE can be used to estimate rationality parameters in route choice framework. In addition, the heterogeneous QRE framework is flexible in incorporating each individual's distinct indifference bands and should be more suitable for route choice modeling.

9.2.3. Boundedly rational multi-modal departure-time and route choices

Most existing dynamic travel behavioral models incorporated bounded rationality into both departure-time and route choices and these two choices are jointly estimated. Mode choice is always treated as a separate decision apart from these two choices. In future, a unifying framework of boundedly rational multi-modal departure-time and route choices should be developed to integrate all travel decisions.

Though bounded rationality is appealing in modeling realistic travel behavior and has attracted an increasing number of researchers, it has not drawn sufficient attentions from practitioners and is ignored in real-world implementation, partly because of its heavy computational burdens caused by indifference bands. Transportation planners should take into account more realistic prediction results within boundedly rational modeling framework while making strategic planning policies in future.

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