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Higher-order closed-form solution for the analysis of laminated composite and sandwich plates based on new shear deformation theories

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ABSTRACT

In the present study, new shear deformation theories (algebraic (ADT), exponential (EDT), hyperbolic (HDT), logarithmic (LDT) and trigonometric (TDT)) were developed to analyze the static, buckling and free vibration responses of laminated composite and sandwich plates using Navier closed form solution technique. The present theories assume parabolic variation of transverse shear stresses through the depth of the plate. Besides, the transverse shear stresses vanish at the top and bottom of the plate surfaces. Thus, the necessity of shear correction factor is evaded. The governing differential equations and boundary conditions are obtained from the virtual work principle. Like FSDT, the present models consist of 5 unknowns. The shear stress parameter m that involves in shear strain function is selected through inverse method. To verify the accuracy and applicability of the present models, numerical comparisons were made with 3D elasticity solutions and existing theories. From the obtained results, it is observed that the proposed shear strain functions have significant effects on structural responses. Also, it is observed that the present theories are more accurate than the renowned theory, for the static, buckling and free vibration analysis of laminated plates.

1. Introduction

Composite materials have made gigantic strides over the past few decades in engineering fields such as aerospace, civil, naval, automotive and many other fields. The composite materials have distinguished characteristics such as high stiffness to weight ratio, high strength to weight ratio, outstanding fatigue strength and have the capability to tailor the lamination scheme according to the specific requirement. It possesses a low value of shear modulus than the homogenous isotropic plates. Consequently, it become much pronounced in the transverse shear deformation. Further, the existence of couplings among extension, shearing, bending, and torsion when the plate subjected to loading. In order to predict them effectively, development of an accurate mathematical model is necessary. The actual behavior of plates can be obtained through three-dimensional (3D) elasticity solution [1,2] at high computational cost and in this approach each lamina is taken as a complete 3D solid. Hence, the complexity of the theory intensifies with the increment of lamina. To overcome these effects, two-dimensional (2D) theories were developed that included equivalent single layer theories (ESL), Layerwise (LW) and Zigzag theories (ZZ) [3–8]. Among the single layer theories, Classical laminated plate theory (CLPT) [9] is the overlook of classical plate theory [10]. In which the transverse shear deformation is not considered and hence, the cross section perpendicular to the reference plane remains straight after deformation. Thus, CLPT limits its applicability to a high aspect ratio of plates and also, it underestimates the static response and overestimates the dynamic response. The first order shear deformation theory (FSDT) [11] is based on Reissner and Mindlin assumptions [12,13] which include the transverse shear effects with a linear variation of transverse shear strain through the plate thickness. Hence, to rectify the impractical behavior of the transverse shear strain/stress an artificial shear correction factor should be multiplied with the shear terms. Further, the shear correction factor dependent on layer orientation, loading conditions and boundary conditions.The FSDT theory widely adopted for laminated composite and FGM plates. Chen et al. [14] achieved nonlinear vibration results for FGM plate with Von Karman assumptions. Lanhe [15] given buckling results in thermal field for FGM plate. Panda et al. [16] studied the influence of hygrothermal effects on free vibration of delaminated plates.

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To overcome the limitations of CLPT and FSDT, several higher order shear deformation theories (HSDTs) were developed in the past few decades by assuming cubic, quadratic or higher variations of thickness coordinate. Basset [17] introduced the displacement field in terms of Taylor series expansion of thickness coordinate. Ambartsumian [18] presented bending results for plates and shells. Lo et al. [19] given a higher order plate theory including stretching effects in the thickness direction with 11 unknowns. Levinson [20], Murthy [21] and Reddy [22] developed the renowned theory. Putcha and Reddy [23] studied the dynamic effects on laminated plate with 11 unknowns. Kant et al. [24] presented a HSDT with 11 unknowns for the transient dynamic analysis using finite element methodology (FEM). Later, Kant and Swaminathan [25] given analytical solutions considering HSDTs with 12 and 9 unknowns. Pradyumna and Bandyopadhyay [26] studied static and dynamic responses of composite shells using higher order theory with 9 unknowns. Neves et al. [27] presented a plate theory with 9 unknowns for FGM plate where the in-plane displacement and transverse displacement components are assumed a cubic and quadratic order of thickness coordinate respectively. Talha and Singh [28] studied the static and vibration responses on functionally graded material (FGM) with 13 unknowns. Natarajan and Ganapathy [29] have presented FE results of a higher order plate theory with 13 unknowns for FGM plates. Though the above Taylor series expansion form of HSDTs yield to predict the structural responses with adequate accuracy, Yet they are physically quite complex to interpret in the formulation. Because, the Taylor series coefficient will generate extra unknown variables.

Several authors overcame the above mentioned complexity by developing shear deformation theories based on various shear strain functions. Levy [30], Stein [31] and Touratier [32] used a sinusoidal shear strain functions. Later on the same shear strain function have been handled by various authors for different theories [33–37]. Soldatos [38] proposed a sinusoidal hyperbolic shear strain function. Karama et al. [39] introduced an exponential function for the static and dynamic analysis of laminated plates. Afterwards, Aydogdu [40] redefined the work of Karama et al. [39]. Further, the same function again overlooked by Mantari et al. [41]. Akavaci [42] obtained analytical solutions for laminated plates by developing two hyperbolic functions. Meiche et al. [43] studied the dynamic analysis of FGM plates by proposing a hyperbolic trigonometric function. Mantari et al. [44,45] developed trigonometric shear deformation theories for plates and shells, afterwards the same theories have been assessed using zigzag theories [46,47]. Further, Mantari and Soares [48,49] have developed Quasi 3D plate theory with 6 unknowns for FGM plate. Neves et al. [50] studied a hyperbolic theory for FGM plate including transverse normal strain effects with 6 unknowns. Further, the same author's [51] have developed a sinusoidal hyperbolic theory for FGM plate with 9 unknowns by mesh free method. Mantari et al. [52] proposed a Quasi 3D trigonometric plate theory with 5 unknowns. Grover et al. [53] assessed the bending and buckling analysis of laminated plates using an inverse trigonometric function. Also, they [54] have examined the static and dynamic responses of laminated plate by proposing trigonometric functions. Thai et al. [55] have given vibration results for FGM plate using a Quasi 3D theory with 5 unknowns. Thai et al. [56] presented a inverse hyperbolic theory based on isogeometric method for laminated plates. Nguyen et al. [57] presented a trigonometric analytical solution for the advanced laminated plate with 5 unknowns. Similarly Belabed et al. [58] also presented a Quasi 3D plate theory for FGM plate using a trigonometric function of Mantari et al. [59] with 5 unknowns. Mantari et al. [60] proposed a quasi 3D plate theory with 4 unknowns for FGM plate. Al Khateeb and Zenkour [61] have developed a 4 unknown plate theory for FGM plate resting on elastic foundation in hygrothermal environment. Mohamed et al. [62] given a four unknown quasi 3D plate theory for FGM plate in hygrothermal environment. Zenkour [63] has given a 4-unknown Quasi 3D plate theory for composite plate resting on elastic foundation. Similarly several shear deformation theories were developed based on different shear strain functions with six unknowns [64,65], five unknowns [66–72] and four unknowns [73–76] for composite and FGM plates. Though the higher order theories have the capability to predict various structural behaviors accurately, still they are unable to account for transverse shear stress continuity at the layer interfaces. In order to include the shear stress continuity ZZ [77–79] and LW [80–84] theories are developed with high computational efforts. However, the continuity of shear stresses can be obtained by integrating the equilibrium equations [22] in HSDTs.

The best of author' s knowledge for the first time this paper presents higher-order closed- form solution for laminated composite and sandwich plates of static, buckling and free vibration analysis using new algebraic (ADT), exponential (EDT), hyperbolic (HDT), logarithmic (LDT) and trigonometric (TDT) shear deformation theories. Using the virtual work principle and calculus of variation the governing differential equations and boundary conditions are derived for the plate structure. A generalized formulation and coding are developed for shear deformation theory that includes shear strain function with five unknowns. The closed form Navier solution method is adopted to obtain the analytical solution for static and dynamic analysis. The shear stress parameter m which involves in shear strain function is optimized through inverse method. The present models represent non-linear variation of transverse shear stresses across the plate thickness. The transverse shear stresses are vanishes at the plate surfaces. Therefore, they neglect the shear correction coefficient. Also, the proposed models represent a non-linear variation of in-plane displacement across the plate thickness. The solution methodology is restricted to simply supported boundary conditions, nevertheless which doesn't carry computational and numerical error. Several numerical examples are conducted considering side to thickness ratio (a/h), length to width ratio (b/a), modulus ratio (E_1/E_2), core to face thickness ratio (t_c/t_f), number of layers (n), layer orientation (θ°) and loading conditions (q). The influences of the proposed shear strain functions on various structural behaviors are studied, and the evaluated results are validated with 3D elasticity solution and available different numerical techniques based shear deformation theories. To demonstrate the efficacy and accuracy of the developed theories global average error percentage with respect to 3D elasticity solution is done which made certain the applicability of the present models. The proposed theories have the potency to predict the structural responses of laminated composite and sandwich plates with adequate accuracy.

2. Theoretical formulation

Consider a rectangular laminated plate with N number of orthotropic layers as shown in Fig. 1. The plate is located in Cartesian coordinates $(x-y-z)$ with length a, width b and total thickness h. The fibers are oriented at θ° angle. The dotted line denotes the reference plane $(z=0)$. The proposed mathematical model can be represented in terms of generalized shear strain function $f(z)$ as

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) - zw_{0,x} + f(z)\phi_x(x,y,t) \\ v(x,y,z,t) &= v_0(x,y,t) - zw_{0,y} + f(z)\phi_y(x,y,t) \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \quad (1)$$

where ",x" and ",y" are the partial derivatives with respect to x and y axis respectively. The in-plane displacement and transverse displacement components are denoted as u, v and w respectively. As well, the reference plane displacements are denoted as u_0, v_0 and

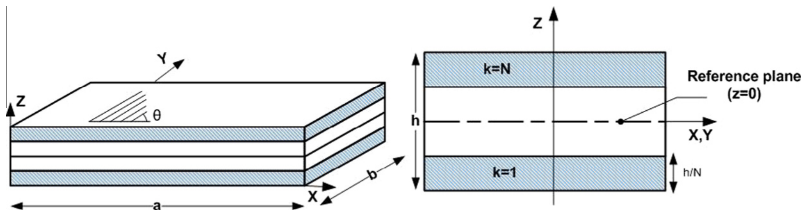

Fig. 1. Schematic diagram of laminated composite plate.

w_0. The rotations about the y and x axis represented as ϕ_x and ϕ_y. The term $f(z)$ assumed as $f(z) = g(z) + \aleph z$. Various proposed shear strain functions $f(z)$ and that includes shear stress parameter m are illustrated in Table 1.

2.1. Strain–displacement relation

The linear strain displacement relations are expressed as follows

$$\begin{aligned}
\varepsilon_{xx} &= u_{0,x} - zw_{0,xx} + [g(z) + \aleph z]\phi_{x,x};\\
\varepsilon_{yy} &= v_{0,y} - zw_{0,yy} + [g(z) + \aleph z]\phi_{y,y};\\
\varepsilon_{xy} &= u_{0,y} + v_{0,x} - 2zw_{0,xy} + [g(z) + \aleph z](\phi_{x,y} + \phi_{y,x});\\
\gamma_{yz} &= (g'(z) + \aleph)\phi_y;\\
\gamma_{xz} &= (g'(z) + \aleph)\phi_x
\end{aligned} \tag{2}$$

here prime denotes the derivative with respect to z.

2.2. Stress–strain relation

The stress strain relation for the kth layer can be expressed as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{3}$$

where $\{\bar{\sigma}\} = \{\sigma_{xx} \;\; \sigma_{yy} \;\; \tau_{xy} \;\; \tau_{yz} \;\; \tau_{xz}\}$ and $\{\bar{\varepsilon}\} = \{\varepsilon_{xx} \;\; \varepsilon_{yy} \;\; \gamma_{xy} \;\; \gamma_{yz} \;\; \gamma_{xz}\}$ are stress and strain vector and $\bar{Q}_{ij}^{(k)}$ are the transformed rigidity matrix of kth layer.

The governing differential equation of the structural system is derived from the virtual work principle Eq. (4) and variation of calculus.

$$\int_{to}^{t} (\delta T - \delta U)dt + \int_{to}^{t} \delta W \, dt = 0 \tag{4}$$

In this present work the potential energy due to external load such as q and $\bar{N}$ assumed as $\delta W = -\delta V$. Thus the kinetic (δT), strain (δU), and potential energy (δV) of the structural system can be expressed as

$$\begin{aligned}
\delta U &= \int_{\Omega} \left\{ \int_{\frac{-h}{2}}^{\frac{h}{2}} \left[\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{xz}\delta\gamma_{xz} \right] dz \right\} dxdy\\
\delta T &= \int_{\Omega} \left\{ \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho[\ddot{u}\delta u_0 + \ddot{v}\delta v_0 + \ddot{w}\delta w_0] dz \right\} dxdy\\
\delta V &= \int_{\Omega} (q + \bar{N})\delta w dxdy
\end{aligned} \tag{5}$$

where $\bar{N} = \bar{N}_{xx}w_{0,xx} + 2\bar{N}_{xy}w_{0,xy} + \bar{N}_{yy}w_{0,yy}$, q denotes the transverse load, $\bar{N}_{xx}$, $\bar{N}_{yy}$ and $\bar{N}_{xy}$ are axial compressive and inplane shear loads. Using Eqs. (5) in (4) the governing differential equations Eq. (6) are obtained after taking integrating by parts. Here the following five equations represents the coefficients of δu_0, δv_0, δw_0, $\delta\phi_y$ and $\delta\phi_x$ respectively.

$$\begin{aligned}
&N_{xx,x} + N_{xy,y} = I_0\ddot{u}_0 - I_1 w_{0,x} + I_3\ddot{\phi}_x;\\
&N_{xy,x} + N_{yy,y} = I_0\ddot{v}_0 - I_1 w_{0,y} + I_3\ddot{\phi}_y;\\
&M_{xx,xx} + 2M_{xy,xy} + M_{yy,yy} + \bar{N} + q = I_1(\ddot{u}_{0,x} + \ddot{v}_{0,y}) - I_2(\ddot{w}_{0,xx} + \ddot{w}_{0,yy})\\
&\qquad + I_4\left(\ddot{\phi}_{x,x} + \ddot{\phi}_{y,y}\right) + I_0\ddot{w}_0;\\
&\aleph(M_{xy,x} + M_{yy,y} - Q_2) + P_{yy,y} - L_2 = I_3\ddot{v}_0 - I_4 w_{0,y} + I_5\ddot{\phi}_y;\\
&\aleph(M_{xx,x} + M_{xy,y} - Q_1) + P_{xx,x} - L_1 = I_3\ddot{u}_0 - I_4 w_{0,x} + I_5\ddot{\phi}_x
\end{aligned} \tag{6}$$

The stress resultants and inertias are expressed as

$$\begin{bmatrix} N_{xx} & M_{xx} & P_{xx} \\ N_{yy} & M_{yy} & P_{yy} \\ N_{xy} & M_{xy} & P_{xy} \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} [1 \; z \; g(z)] dz$$

$$\begin{bmatrix} Q_2 & L_2 \\ Q_1 & L_1 \end{bmatrix} = \int_{\frac{-h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix} [1 \; g'(z)] dz$$

$$[I_0 \; I_1 \; I_2 \; I_3 \; I_4 \; I_5] = \int_{\frac{-h}{2}}^{\frac{h}{2}} \rho^k \left[1 \; z \; z^2 \; (g(z) + \aleph z) \; z(g(z) + \aleph z) \; (g(z) + \aleph z)^2 \right] dz$$

Using the following expressions, the differential equations Eqs. (6) are defined in terms of displacements and rotations.

Table 1
Proposed shear strain functions.

S.No	$g(z)$	$\aleph$	m	Abbreviation	Explanation
1	$\frac{mz}{h\left(\frac{m^2z^2}{h^2}+1\right)}$	$-\frac{8mh^2}{(m^3h^3+8h^3)}$	2	ADT	Algebraic
2	$\left(\frac{mz}{h}\right)^3 e^{(2\sqrt{\pi})}$	$-\frac{3e^{2\sqrt{\pi}}m^3}{4h}$	0.95	EDT	Exponential
3	$tanh\left(\frac{mz}{h}\right)$	$-\frac{m}{h}sech^2\left(\frac{m}{2}\right)$	2.5	HDT	Hyperbolic
4	$\left(\frac{z}{h}\right)^2 mlog\left[e^{\left(\frac{z}{h}\right)}\right]$	$-\frac{m^2}{h}\left\{log\left[e^{\left(\frac{1}{2}\right)}\right] + \frac{1}{4}\right\}$	1.04	LDT	Logarithmic
5	$sin\left(\frac{mz}{h}\right)cos\left(\frac{mz}{h}\right)$	$-\left[\frac{m}{h}cos(m)\right]$	2.5	TDT	Trigonometric

$$[A_{ij}\ B_{ij}\ D_{ij}\ E_{ij}\ F_{ij}\ H_{ij}] = \int_{\frac{-h}{2}}^{\frac{h}{2}} \left[Q_{ij}^{(k)}\right] \left[1\ z\ z^2\ g(z)\ zg(z)\ (g(z))^2\right] dz \quad (i,j = 1,2,4,5,6)$$

$$[R_{ij}\ S_{ij}] = \int_{\frac{-h}{2}}^{\frac{h}{2}} \left[Q_{ij}^{(k)}\right] \left[g'(z)\ (g'(z))^2\right] dz \quad (i,j = 4,5)$$

Hence Eqs. (6) may be defined in terms of generalized displacement considering the aforementioned expressions:

$$\begin{aligned}
&A_{11}u_{0,xx} + B_{11}(\aleph\phi_{x,xx} - w_{0,xxx}) + E_{11}\phi_{x,xx} + A_{12}v_{0,xy} + B_{12}(\aleph\phi_{y,xy} \\
&\quad - w_{0,xyy}) + E_{12}\phi_{y,xy} + A_{66}(u_{0,yy} + v_{0,xy}) + B_{66}[\aleph(\phi_{x,yy} + \phi_{y,xy}) \\
&\quad - 2w_{0,xyy}] + E_{66}(\phi_{x,yy} + \phi_{y,xy}) \\
&\quad = I_0\ddot{u}_0 - I_1 w_{0,x} + I_3\ddot{\phi}_x
\end{aligned}$$

$$\begin{aligned}
&A_{12}u_{0,xy} + B_{12}(\aleph\phi_{x,xy} - w_{0,xxy}) + E_{12}\phi_{x,xy} + A_{22}v_{0,yy} + B_{22}(\aleph\phi_{y,yy} \\
&\quad - w_{0,yyy}) + E_{22}\phi_{y,yy} + A_{66}(u_{0,xy} + v_{0,xx}) + B_{66}[\aleph(\phi_{x,xy} + \phi_{y,xx}) \\
&\quad - 2w_{0,xxy}] + E_{66}(\phi_{x,xy} + \phi_{y,xx}) \\
&\quad = I_0\ddot{v}_0 - I_1 w_{0,y} + I_3\ddot{\phi}_y
\end{aligned}$$

$$\begin{aligned}
&B_{11}u_{0,xxx} + D_{11}(\aleph\phi_{x,xxx} - w_{0,xxxx}) + F_{11}\phi_{x,xxx} + B_{12}(v_{0,xxy} + u_{0,xyy}) \\
&\quad + D_{12}[\aleph(\phi_{y,xxy} + \phi_{x,xyy}) - 2w_{0,xxyy}] + F_{12}(\phi_{y,xxy} + \phi_{x,xyy}) + B_{22}v_{0,yyy} \\
&\quad + D_{22}(\aleph\phi_{y,yyy} - w_{0,yyyy}) + F_{22}\phi_{y,yyy} + 2B_{66}(u_{0,xyy} + v_{0,xxy}) + q \\
&\quad + \overline{N}_{xx}w_{0,xx} + 2\overline{N}_{xy}w_{0,xy} + \overline{N}_{yy}w_{0,yy} \\
&\quad = I_1(\ddot{u}_{0,x} + \ddot{v}_{0,y}) - I_2(\ddot{w}_{0,xx} + \ddot{w}_{0,yy}) + I_4(\ddot{\phi}_{x,x} + \ddot{\phi}_{y,y}) + I_0\ddot{w}_0
\end{aligned}$$

$$\begin{aligned}
&(\aleph B_{11} + E_{11})u_{0,xx} + (\aleph D_{11} + F_{11})(\aleph\phi_{x,xx} - w_{0,xxx}) + (\aleph F_{11} + H_{11})\phi_{x,xx} \\
&\quad + (\aleph B_{12} + E_{12})v_{0,xy} + (\aleph D_{12} + F_{12})(\aleph\phi_{y,xy} - w_{0,xyy}) \\
&\quad + (\aleph F_{12} + H_{12})\phi_{y,xy} + (\aleph B_{66} + E_{66})(u_{0,yy} - v_{0,xy}) \\
&\quad + (\aleph D_{66} + F_{66})[\aleph(\phi_{x,yy} + \phi_{y,xy}) - 2w_{0,xyy}] \\
&\quad + (\aleph F_{66} + H_{66})(\phi_{x,yy} - \phi_{y,xy}) - [\aleph^2 A_{55} + 2\aleph R_{55} + S_{55}]\phi_x = I_3\ddot{u}_0 \\
&\quad - I_4 w_{0,x} + I_5\ddot{\phi}_x
\end{aligned}$$

$$\begin{aligned}
&(\aleph B_{12} + E_{12})u_{0,xy} + (\aleph D_{12} + F_{12})(\aleph\phi_{x,xy} - w_{0,xxy}) + (\aleph F_{12} + H_{12})\phi_{x,xy} \\
&\quad + (\aleph B_{22} + E_{22})v_{0,yy} + (\aleph D_{22} + F_{22})(\aleph\phi_{y,yy} - w_{0,yyy}) \\
&\quad + (\aleph F_{22} + H_{22})\phi_{y,yy} + (\aleph B_{66} + E_{66})(u_{0,xy} + v_{0,xx}) \\
&\quad + (\aleph D_{66} + F_{66})[\aleph(\phi_{x,xy} + \phi_{y,xx}) - 2w_{0,xxy}] \\
&\quad + (\aleph F_{66} + H_{66})(\phi_{x,xy} - \phi_{y,xx}) - [\aleph^2 A_{44} + 2\aleph R_{44} + S_{44}]\phi_y = I_3\ddot{v}_0 \\
&\quad - I_4 w_{0,y} + I_5\ddot{\phi}_y
\end{aligned} \tag{7}$$

This work is corned with cross-ply laminated plates. Therefore, the following stiffness components are not considered in the above equations.

$$A_{i6} = B_{i6} = D_{i6} = E_{i6} = F_{i6} = H_{i6} = A_{45} = R_{45} = S_{45} = 0 \quad (i = 1,2)$$

3. Navier solution

The analytical solutions are obtained by imposing the simply supported boundary conditions Eq. (8) at the edges of the rectangular cross-ply plates.

$$\begin{aligned}
v_0 = w_0 = \phi_y = N_{xx} = M_{xx} = 0 \ at\ x = 0, a \\
u_0 = w_0 = \phi_x = N_{yy} = M_{yy} = 0 \ at\ y = 0, b
\end{aligned} \tag{8}$$

We consider the Navier solutions Eq. (9) in the form of double Fourier series for the five unknowns $(u_0, v_0, w_0, \phi_x, \phi_y)$ that satisfy the above stated boundary conditions Eq. (8) and differential equation Eq. (6).

$$\begin{Bmatrix} u_0(x,y,t) \\ v_0(x,y,t) \\ w_0(x,y,t) \\ \phi_x(x,y,t) \\ \phi_y(x,y,t) \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn}e^{i\omega t}cos(\alpha x)sin(\beta y) \\ V_{mn}e^{i\omega t}sin(\alpha x)cos(\beta y) \\ W_{mn}e^{i\omega t}sin(\alpha x)sin(\beta y) \\ X_{mn}e^{i\omega t}cos(\alpha x)sin(\beta y) \\ Y_{mn}e^{i\omega t}sin(\alpha x)cos(\beta y) \end{Bmatrix} \tag{9}$$

where $\alpha = m\pi/a,\ \beta = n\pi/b$. Using the Navier solutions in Eq. (7) a set of equations are achieved in terms of $U_{mn}, V_{mn}, W_{mn}, X_{mn}, Y_{mn}$ parameters.

3.1. Static analysis

Using the Navier solution in Eq. (7) the following matrix form is obtained for the static analysis of rectangular cross-ply laminated plates under consideration of transverse load (q). Hence $\overline{N}$ and inertia terms are neglected from Eq. (7).

$$[\overline{K}]\{\chi\} = \{q\} \tag{10}$$

here

$$\{\chi\} = \{U_{mn}\ V_{mn}\ W_{mn}\ X_{mn}\ Y_{mn}\}^T, \quad q = \{0\ 0\ q_{mn}\ 0\ 0\}^T$$

The transverse load q taken in terms of double Fourier series as follows:

$$q = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn} sin(\alpha x) sin(\beta y)$$

where

$$q_{mn} = \begin{cases} q_0; & for\ sinusoidal\ load\ (SSL) \\ 16q_0/(\pi^2 mn); & for\ uniformly\ distributed\ load\ (UDL) \end{cases}$$

the elements of coefficients $[\overline{K}]$ are defined in Appendix.

3.2. Eigenvalue problem for buckling and free vibration analysis

To evaluate the buckling analysis of laminated plates uni-axial compressive load is considered. Hence $\overline{N}_{yy}, \overline{N}_{xy}$, q and inertia terms are neglected for a bucking problem. The following eigenvalue equations is obtained after using Navier solutions Eq. (9) in Eq. (6).

$$\left[[\overline{K}] - \lambda^2[G]\right]\{\chi\} = \{0\} \tag{11}$$

As well, the eigenvalue equation Eq. (12) is obtained for the free vibration analysis neglecting $\overline{N}_{xx}, \overline{N}_{yy}, \overline{N}_{xy}, q$ terms.

$$\left[[\overline{K}] - \omega^2[M]\right]\{\chi\} = \{0\} \tag{12}$$

where $[G]$ denotes the geometric matrix due to the uni-axial load, λ denotes the buckling parameter, ω denotes the frequency parameter. The non zero terms of mass matrix $[M]$ are

$$M_{11} = I_0;\ M_{13} = -I_1\alpha;\ M_{14} = I_3;\ M_{22} = I_0;\ M_{23} = -I_1\beta;\ M_{25} = I_3;$$
$$M_{33} = I_0 + I_2(\alpha^2 + \beta^2);\ M_{34} = -I_4\alpha;\ M_{35} = -I_4\beta;\ M_{44} = I_5;\ M_{55} = I_5$$

4. Numerical results and discussion

An original generalized program is made on MATLAB platform for static and dynamic analysis of laminated composite and sandwich plate. To show the efficacy of the present models several numerical examples are carried and validated with those of other shear deformation theories. The parameter m values for each shear strain function are optimized through inverse method at the post process [40]. The parameter m value selected in such a way they give relatively near results to 3D elasticity solution [85]. The optimized values of m for each theory are given in Table 1. To show the

Table 2
Elastic constants.

Elastic constants	Unit	EC1	EC2	EC3	EC4
E_2	GPa	E_2	E_2	–	10.34
E_1	GPa	$25E_2$	open	$0.543E_1$	131
E_3	GPa	E_2	E_2	E_1	6.9
G_{12}	GPa	$0.5E_2$	$0.6E_2$	$0.2629E_1$	6.9
G_{13}	GPa	$0.5E_2$	$0.6E_2$	$0.1599E_1$	6.2
G_{23}	GPa	0.2	0.5	$0.2668E_1$	6.9
ν_{12}	–	0.25	0.25	0.3	0.22
ρ	kg/m^3	1	1	–	1627

efficacy of the proposed models error percentage Eq. (13) with respect to 3D elasticity solution is calculated.

$$Error = [Present\ result - 3D\ Elasticity/3D\ Elasticity] \times 100 \qquad (13)$$

The static and dynamic analysis of laminated composites and sandwich plates are assessed using the elastic constants given in Table 2.

4.1. Static analysis

The following non-dimensional transverse deflection and normal and shear stresses are used for the static analysis of laminated composites and sandwich plates.

$$w = w_0\left(\frac{a}{2},\frac{b}{2},0\right)\left(\frac{100E_2h^3}{a^4q_0}\right); \qquad (14)$$
$$[\overline{\sigma}_{xx}\overline{\sigma}_{yy}\overline{\tau}_{xy}\overline{\tau}_{xz}\overline{\tau}_{yz}] = \left[\frac{h}{a}\sigma_{xx}\frac{h}{a}\sigma_{yy}\frac{h}{a}\tau_{xy}\tau_{xz}\tau_{yz}\right]\left(\frac{h}{aq_0}\right);$$

4.1.1. Four layered [0°/90°/90°/0°] square laminated plate under SSL

In this example, a four layered square cross-ply [0°/90°/90°/0°] laminated plate under SSL is studied. The non-dimensional deflection and stresses are computed using the elastic constants EC1 and Eq. (14). By varying the side to thickness ratio a/h = 4 to100 (thick to thin plate) the static analysis is performed and the results are tabulated in Table 3.

The evaluated results are compared with 3D elasticity exact solution of Pagano and Hatfield [85]. Furthermore, the analytical solution given by Reddy [22] and Karama et al. [39] are also considered for validation. To show the accuracy and reliability of the present models average and global error percentage calculated with respect to 3D elasticity solution [85]. It is seen from the Table 3 the present ADT globally gives 4.00%, 6.77% of EDT, HDT gives 3.91%, 6.95% of LDT, 3.62% of TDT, 4.8% of Karama et al. [39], 7.25 % of Reddy [22]. It is clearly visible that the present TDT provides improved predictions compared to other proposed models and existing theories [39,22]. Further the present HDT and ADT also gives improved results than the Karama et al. [39] and Reddy [22]. The present EDT and LDT are less accurate than Karama et al. [39], however they perform better than Reddy [22]. Also, it is observed that TDT, HDT and ADT performs very well for a thick laminated plate.

The variation of normal stress ($\overline{\sigma}_{xx}$) and transverse shear stress ($\overline{\tau}_{yz}$) of the present models are plotted for side thickness ratio 10. Fig. 2 represents the normal distribution of the present models along with Reddy model [22]. The TDT, HDT and ADT give enhanced response than Reddy [22] and EDT and LDT are in excellent agreement with Reddy [22]. Further, the non-linear variation of present transverse shear stresses is plotted and compared with the existing theory [22]. Thus, the present models confirm the sur-

Table 3
Four layered [0°/90°/90°/0°] square laminated plate under SSL.

a/h	Source	$\overline{w}(\frac{a}{2},\frac{b}{2},0)$	Error%	$\overline{\sigma}_{xx}(\frac{a}{2},\frac{b}{2},\frac{h}{2})$	Error%	$\overline{\sigma}_{yy}(\frac{a}{2},\frac{b}{2},\frac{h}{4})$	Error%	$\overline{\tau}_{xy}(0,0,\frac{h}{2})$	Error%	$\overline{\tau}_{xz}(0,\frac{b}{2},0)$	Error%	$\overline{\tau}_{yz}(\frac{a}{2},0,0)$	Error%	Avg%
4	Present-ADT	1.9254	−1.46	0.7269	0.96	0.6384	−3.72	0.0473	0.63	0.2483	13.40	0.2679	−7.95	4.69
	Present-EDT	1.9290	−1.28	0.6497	−9.77	0.6411	−3.30	0.0439	−6.59	0.2084	−4.85	0.2436	−16.29	7.01
	Present-HDT	1.9255	−1.46	0.7255	0.76	0.6381	−3.76	0.0472	0.44	0.2457	12.17	0.2659	−8.63	4.54
	Present-LDT	1.9278	−1.34	0.6530	−9.30	0.6429	−3.03	0.0441	−6.19	0.2090	−4.57	0.2427	−16.60	6.84
	Present-TDT	1.9235	−1.56	0.7188	−0.17	0.6369	−3.93	0.0468	−0.45	0.2359	7.72	0.2589	−11.02	4.14
	Exact [85]	1.954	0.0	0.72	0.00	0.663	0.00	0.047	0.00	0.219	0.00	0.291	0.00	0.00
	Karama [39]*	1.919	−1.8	0.699	−2.92	0.636	−4.07	0.0459	−2.34	0.226	3.20	0.226	−22.34	6.11
	Reddy [22]*	1.893	−3.1	0.665	−7.64	0.632	−4.68	0.044	−6.38	0.206	−5.94	0.239	−17.87	7.60
10	Present-ADT	0.7286	−1.94	0.5581	−0.17	0.3948	−1.55	0.0274	−1.99	0.3263	8.41	0.1748	−10.80	4.14
	Present-EDT	0.7221	−2.81	0.5418	−3.08	0.3923	−2.17	0.0268	−4.29	0.2672	−11.24	0.1553	−20.79	7.40
	Present-HDT	0.7284	−1.96	0.5578	−0.22	0.3947	−1.56	0.0274	−2.04	0.3226	7.18	0.1733	−11.56	4.09
	Present-LDT	0.7192	−3.21	0.5426	−2.93	0.3911	−2.46	0.0268	−4.40	0.2654	−11.84	0.1537	−21.58	7.74
	Present-TDT	0.7271	−2.14	0.5563	−0.48	0.3942	−1.70	0.0274	−2.32	0.3088	2.60	0.1680	−14.26	3.92
	Exact [1]	0.743	0.00	0.559	0.00	0.401	0.00	0.028	0.00	0.301	0.00	0.196	0.00	0.00
	Karama [39]*	0.724	−2.56	0.553	−1.07	0.393	−2.00	0.027	−3.57	0.294	−2.33	0.163	−16.84	4.73
	Reddy [22]*	0.715	−3.77	0.546	−2.33	0.389	−2.99	0.027	−3.57	0.264	−12.29	0.153	−21.94	7.81
20	Present-ADT	0.5102	−1.31	0.5425	−0.09	0.3064	−0.52	0.0230	−0.05	0.3517	7.21	0.1401	−10.22	3.23
	Present-EDT	0.5081	−1.73	0.5382	−0.88	0.3054	−0.85	0.0228	−0.80	0.2863	−12.72	0.1246	−20.14	6.19
	Present-HDT	0.5102	−1.32	0.5425	−0.10	0.3064	−0.52	0.0230	−0.07	0.3476	5.98	0.1388	−11.00	3.17
	Present-LDT	0.5071	−1.91	0.5385	−0.83	0.3049	−1.00	0.0228	−0.85	0.2837	−13.52	0.1235	−20.84	6.49
	Present-TDT	0.5097	−1.40	0.5421	−0.17	0.3062	−0.59	0.0230	−0.16	0.3325	1.37	0.1346	−13.70	2.90
	Exact [1]	0.517	0.00	0.543	0.00	0.308	0.00	0.023	0.00	0.328	0.00	0.156	0.00	0.00
	Karama [39]*	0.509	−1.55	0.541	−0.37	0.306	−0.65	0.023	0.00	0.316	−3.66	0.131	−16.03	3.71
	Reddy [22]*	0.506	−2.13	0.539	−0.74	0.304	−1.30	0.023	0.00	0.283	−13.72	0.123	−21.15	6.51
100	Present-ADT	0.4345	−1.03	0.5388	−0.03	0.2709	−1.84	0.0214	−2.95	0.3617	7.32	0.1260	−10.62	3.96
	Present-EDT	0.4344	−1.05	0.5387	−0.06	0.2709	−1.86	0.0213	−2.98	0.2938	−12.82	0.1124	−20.27	6.51
	Present-HDT	0.4345	−1.03	0.5388	−0.03	0.2709	−1.84	0.0214	−2.95	0.3575	6.08	0.1249	−11.39	3.89
	Present-LDT	0.4343	−1.06	0.5387	−0.06	0.2709	−1.87	0.0213	−2.98	0.2908	−13.70	0.1116	−20.88	6.76
	Present-TDT	0.4345	−1.03	0.5388	−0.03	0.2709	−1.84	0.0214	−2.95	0.3418	1.43	0.1212	−14.05	3.56
	Exact [1]	0.439	0.00	0.539	0.00	0.276	0.00	0.022	0.00	0.337	0.00	0.141	0.00	0.00
	Karama [39]*	0.435	−0.91	0.538	−0.19	0.27	−2.17	0.021	−4.55	0.324	−3.86	0.118	−16.31	4.66
	Reddy [22]*	0.434	−1.14	0.538	−0.19	0.27	−2.17	0.021	−4.55	0.29	−13.95	0.112	−20.57	7.09

* Analytical solution-HSDT.

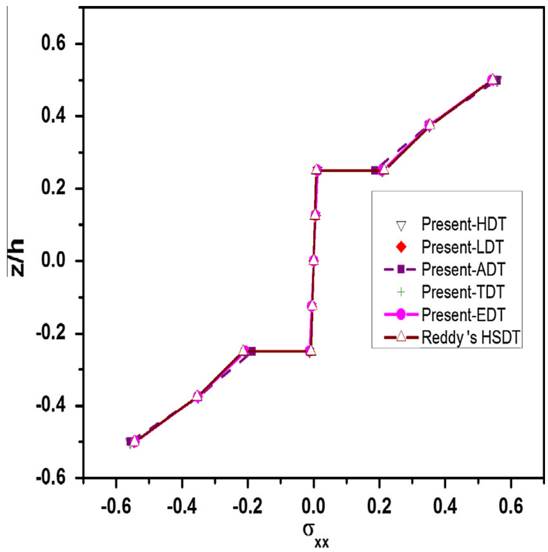

Fig. 2. Normal stress ($\overline{\sigma}_{xx}$).

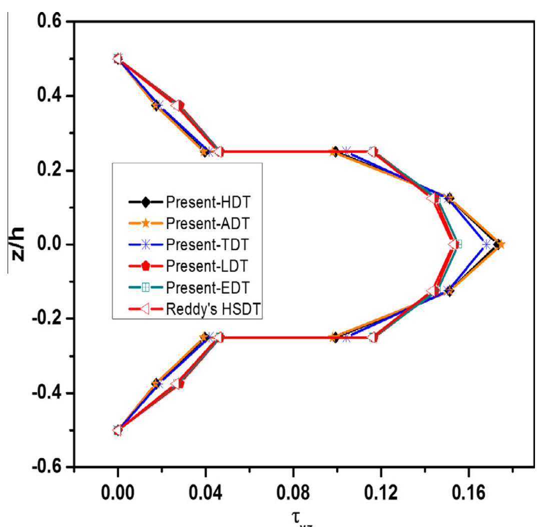

Fig. 3. Transverse shear stress ($\overline{\tau}_{yz}$).

face free boundary conditions. From Fig. 3 it is observed that among all proposed theories ADT provides more accurate results. Also, it clearly reveals that the present models produce better prediction as compared to that of Reddy [22].

4.1.2. Three layered [0°/90°/0°] laminated plate

A three layered laminated cross-ply [0/90/0] square plate is subjected to SSL load. The static analysis is carried out using the non-dimensional form is given in Eq. (14) and elastic constants EC1. From Table 4 it can be seen that present HDT and ADT are in absolute agreement with Mantari et al. [45], and more accurate than other compared results [39,22]. The present TDT less accurate than the Mantari et al. [45], however, yields to produce better results than Karama et al. [39] and Reddy [22]. The EDT and LDT more accurate than [22] nevertheless, less accurate than Manatri et al. [45] and Karama et al. [39]. The above mentioned same configured plate is subjected to UDL. The non-dimensional deflection is calculated for $a/h = 4,10,20,50,100$ of developed shear strain functions and they have shown in Table 6 along with existing theories [86,87,22]. Sheik and Chakrabarti [86] have given finite element results for Reddy's theory [22] using non-conforming element with 42 unknowns per node. Singh et al. [87] given zigzag theory using FEM. It is noted that ADT yields to stand better than other proposed and published results [86,87,22]. HDT and TDT also leads to give better results than the existing results.

The static responses of the present theories for rectangular laminated plate $b = 3a$ under SSL is given in Table 5 along with 3D elasticity solution [85] and published results [39,32,22]. The present ADT leads to giving better results than the existing and other proposed shear deformation theories. The HDT and TDT also give better results than published results [39,32,22], however, less accurate than ADT. From the whole discussion, it can says that ADT, HDT and TDT yields to perform very well for laminated composite plates especially for thick plate whereas EDT and LDT are in well agreement with the published results.

4.1.3. Asymmetric two layered laminated [0°/90°] plate under SSL

In this section, the static analysis is performed for two layered [0°/90°] laminated plate under SSL. The plate edges are simply supported. The influence of side to thickness ratios on maximum central deflections (w) are examined using elastic constants EC1 and non-dimensional form given in Eq. (14). The results are related to the elasticity solution [1] in Fig. 4. It is clear that the present models give the same response as discussed in the symmetric laminated plates of static response.

4.1.4. Three layered sandwich plate under UDL

For this assessment a three layered sandwich plate [0°/C/0°] with side to thickness ratio 10 is taken. The non-dimensional central deflection and normal and shear stresses are assessed using Eq. (15).

$$\begin{aligned}
&w = w_0\left(\frac{a}{2},\frac{a}{2},0\right)\frac{0.999781}{hq}; \quad \overline{\sigma}^1_{xx} = \sigma^1_{xx}\left(\frac{a}{2},\frac{b}{2},-\frac{h}{2}\right)\frac{1}{q};\\
&\overline{\sigma}^2_{xx} = \sigma^1_{xx}\left(\frac{a}{2},\frac{b}{2},-\frac{2h}{5}\right)\frac{1}{q}; \quad \overline{\sigma}^3_{xx} = \sigma^2_{xx}\left(\frac{a}{2},\frac{b}{2},-\frac{2h}{5}\right)\frac{1}{q};\\
&\overline{\sigma}^1_{yy} = \sigma^1_{yy}\left(\frac{a}{2},\frac{b}{2},-\frac{h}{2}\right)\frac{1}{q}; \quad \overline{\sigma}^2_{yy} = \sigma^1_{yy}\left(\frac{a}{2},\frac{b}{2},-\frac{2h}{5}\right)\frac{1}{q};\\
&\overline{\sigma}^3_{yy} = \sigma^2_{yy}\left(\frac{a}{2},\frac{b}{2},-\frac{2h}{5}\right)\frac{1}{q};
\end{aligned} \tag{15}$$

The middle layer of the core has the thickness of 0.8 h whereas the outer layers possess 0.1 h thickness. Here h denotes the total plate thickness. The following elastic properties are used for the core layer whereas the elastic properties of face layers are obtained from the relation given in Eq. (17).

$$\overline{Q}_{core} = \begin{bmatrix} 0.999781 & 0.231192 & 0 & 0 & 0\\ 0.231192 & 0.524886 & 0 & 0 & 0\\ 0 & 0 & 0.262931 & 0 & 0\\ 0 & 0 & 0 & 0.266810 & 0\\ 0 & 0 & 0 & 0 & 0.159914 \end{bmatrix} \tag{16}$$

$$\overline{Q}_{face} = R\overline{Q}_{core} \tag{17}$$

The influence of parameter R on the static analysis of sandwich plate is investigated and tabulated in Table 7 along with Srinivas [81] and mesh free methodology of Xiang et al. [88] for Touraiter [32], Karama et al. [39] and Levinson [20]. To show the performance of the static analysis of present models on sandwich plate global error percentage with respect to the exact solution Srinivas [81] is calculated. Table 7 shows that TDT leads to generate more accurate results as compared to the other shear deformation theories. The HDT and ADT also giveenhanced results as compared to

Table 4
Three layered [0°/90°/0°] laminated square plate under SSL.

a/h	Source	$\overline{w}(\frac{a}{2},\frac{b}{2},0)$	$\overline{\sigma}_{xx}(\frac{a}{2},\frac{b}{2},\frac{h}{2})$	$\overline{\sigma}_{yy}(\frac{a}{2},\frac{b}{2},\frac{h}{4})$	$\overline{\tau}_{xy}(0,0,\frac{h}{2})$	$\overline{\tau}_{xz}(0,\frac{b}{2},0)$	$\overline{\tau}_{yz}(\frac{a}{2},0,0)$
4	Present-ADT	1.9519	0.8087	0.5009	0.0532	0.2414	0.20045
	Present-EDT	1.9606	0.7197	0.5117	0.0498	0.2050	0.18742
	Present-HDT	1.9505	0.8066	0.5010	0.0530	0.2386	0.19919
	Present-LDT	1.9562	0.7235	0.5115	0.0499	0.2050	0.1864
	Present-TDT	1.9449	0.7970	0.5012	0.0525	0.2289	0.19488
	Exact [85]	2.0060	0.7550	0.5560	0.0505	0.2820	0.2170
	Mantari et al. [45]	1.9434	0.8230	0.4970	0.0536	0.2450	0.2010
	Karama et al. [39]	1.9440	0.7750	0.5020	0.0516	0.2200	0.1910
	Reddy [22]	1.9218	0.7340	–	–	–	0.1830
10	Present-ADT	0.7321	0.5847	0.2755	0.0286	0.3059	0.1140
	Present-EDT	0.7197	0.5652	0.2715	0.0277	0.2477	0.1050
	Present-HDT	0.7312	0.5842	0.2751	0.0285	0.3017	0.1132
	Present-LDT	0.7163	0.5658	0.2704	0.0277	0.2456	0.1040
	Present-TDT	0.7273	0.5818	0.2738	0.0284	0.2868	0.1106
	Exact [85]	0.7405	0.5900	0.2880	0.0289	0.357	0.123
	Mantari et al. [45]	0.7342	0.5880	0.2760	0.0288	0.314	0.115
	Karama et al. [39]	0.7230	0.5760	0.2720	0.0281	0.272	0.108
	Reddy [22]	0.7125	0.5680	–	–	–	0.103
100	Present-ADT	0.4344	0.5392	0.1807	0.0214	0.3275	0.0808
	Present-EDT	0.4343	0.5390	0.1806	0.0214	0.2619	0.0757
	Present-HDT	0.4344	0.5392	0.1807	0.0214	0.3228	0.0804
	Present-LDT	0.4342	0.5390	0.1806	0.0214	0.2590	0.0752
	Present-TDT	0.4344	0.5391	0.1807	0.0214	0.3060	0.0788
	Exact [85]	–	0.5390	0.1810	0.0213	0.3950	0.0830
	Mantari et al. [45]	0.4353	0.5390	0.1810	0.0214	0.3370	0.0810
	Karama et al. [39]	0.4350	0.5380	0.1800	0.0213	0.2890	0.0780
	Reddy [22]	0.4342	0.5390	–	–	–	0.0750

Table 5
Three layered [0°/90°/0°] laminated rectangular plate under SSL.

a/h	Source	$\overline{w}(\frac{a}{2},\frac{b}{2},0)$	$\overline{\sigma}_{xx}(\frac{a}{2},\frac{b}{2},\frac{h}{2})$	$\overline{\sigma}_{yy}(\frac{a}{2},\frac{b}{2},\frac{h}{4})$	$\overline{\tau}_{xy}(0,0,\frac{h}{2})$	$\overline{\tau}_{xz}(0,\frac{b}{2},0)$	$\overline{\tau}_{yz}(\frac{a}{2},0,0)$
4	Present-ADT	2.7056	1.1461	0.1041	0.0279	0.3275	0.0371
	Present-EDT	2.7062	1.0212	0.1052	0.0266	0.2770	0.0359
	Present-HDT	2.7025	1.1426	0.1040	0.0278	0.3236	0.0369
	Present-LDT	2.6997	1.0263	0.1050	0.0266	0.2771	0.0357
	Present-TDT	2.6899	1.1275	0.1038	0.0275	0.3099	0.0363
	Exact [1]	2.8200	1.1000	0.1190	0.0281	0.3870	0.0334
	Karama [39]	2.6838	1.0970	0.1040	0.0272	0.2980	0.0360
	Touratier [32]	2.6657	1.0670	0.1030	0.0268	0.2850	0.0360
	Reddy [22]	2.6411	1.0360	0.1030	0.0260	0.2720	0.0350
10	Present-ADT	0.8892	0.7147	0.0408	0.0119	0.3588	0.0180
	Present-EDT	0.8718	0.6897	0.0402	0.0116	0.2898	0.0173
	Present-HDT	0.8879	0.7139	0.0407	0.0119	0.3538	0.0179
	Present-LDT	0.8673	0.6899	0.040	0.012	0.287	0.017
	Present-TDT	0.8825	0.7105	0.0405	0.0118	0.3360	0.0177
	Exact [1]	0.9190	0.7250	0.0435	0.0123	0.4200	0.0152
	Karama [39]	0.8768	0.7040	0.0400	0.0117	0.3190	0.0180
	Touratier [32]	0.8698	0.6980	0.0400	0.0116	0.3020	0.0170
	Reddy [22]	0.8622	0.6920	0.0400	0.0120	0.2860	0.0170

Table 6
Three layered [0°/90°/0°] laminated square plate under UDL.

a/h	ADT	EDT	HDT	LDT	TDT	Elasticity [1]	Ref [86][a]	Singh et al. [87][c]	Reddy [22][b]
4	2.9734	2.9655	2.9709	2.9557	2.9583	3.0416	2.9093	2.9096	2.6596
10	1.121	1.1007	1.1197	1.0957	1.1132	1.1533	1.091	1.0981	1.0219
20	0.785	0.7788	0.7846	0.7773	0.7827	–	–	0.7785	0.7573
50	0.6853	0.6842	0.6852	0.6840	0.6849	–	–	0.6843	0.6807
100	0.6708	0.6706	0.6708	0.6705	0.6707	0.6712	0.6708	0.6706	0.6697

[a] FEM for Reddy's HSDT.
[b] FEM for ZZ.
[c] Analytical solution for Reddy's FSDT.

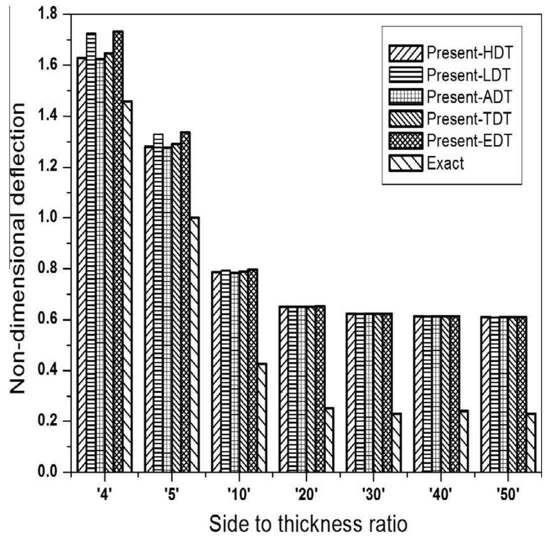

Fig. 4. Non-dimensional deflection for asymmetric laminated [0°/90°] square plate under SSL.

the published results [88]. The LDT and EDT also perform better than the available shear deformation theories. From the whole discussion of this analysis, it is observed that all proposed theories predict the static response of sandwich plate more accurately as compared to the existed results [88].

4.2. Buckling analysis

4.2.1. Symmetric laminated plate under uniaxial load

In this part symmetric three, five, and nine layered cross-ply laminated plates are investigated under uniaxial load. The plate edges are simply supported with side to thickness parameter of 10. The buckling parameter $\overline{\lambda}_{cr} = \lambda a^2/E_2h^3$ is analyzed using elastic constants EC2. The influence of modulus rigidity (E_1/E_2) is assessed against n for the present theories and comparisons made for the obtained results with 3D elasticity solution [2] and available shear deformation theories [54,23] as shown in Table 8. The global average error percentage with respect to elasticity solution is evaluated to show the accuracy of the present theories. The present ADT possess 1.66%, 1.7% of EDT, HDT possess 1.63%, 1.19% of LDT, 1.53% of TDT, 1.66% of Grover et al. [54] and 1.93% of Reddy and Putcha [23]. It is clear that the present LDT predicts the buckling parameter for laminated plate very proficiently than other available and proposed theories [54,23]. The TDT and HDT give improved results next to LDT. The ADT leads to giving more accurate results than Grover et al. [54] and Putcha and Reddy [23]. The EDT lesser than the above mentioned theories, nevertheless better than Putcha and Reddy [23]. It is also revealed that the buckling parameter increases when the modular ratio increases.

4.2.2. Three layered sandwich plate [0°/C/0°] under uniaxial load

A three layered sandwich plate [0°/C/0°] under uniaxial load with a/h ratio 10 is chosen for this examination. The thickness of outer skins are 0.1 h whereas the inner core is 0.8 h. The elastic constants EC3 is used for the inner core. The elastic constants of outer skins are obtained through EC3 model with a multiplication of parameter R in the elastic moduli. Here the parameter R ranging from 1 to 10. The critical buckling parameter $\lambda_{cr} = 12N_{cr}a^2/\pi^2E_1h^2$ for present theories are analyzed and related to exact solution [89], zigzag theory of Di Scuiva [90,91] and analytical solution of Grover et al. [54]. The present EDT yields to give more accurate results than the HSDTs. Next to EDT, the LDT also gives accurate results than other shear deformation theories [54]. From the Table 9 it confirms that the proposed models provide better responses for sandwich plate than Grover et al. [54], however as expected lesser than the zigzag theories [90,91].

Table 7
Three layered sandwich plate [0°/C/0°] under UDL.

R	Source	$\overline{w}$	Diff %	$\overline{\sigma}^1_{xx}$	Diff %	$\overline{\sigma}^2_{xx}$	Diff %	$\overline{\sigma}^3_{xx}$	Diff %	$\overline{\sigma}^1_{yy}$	Diff %	$\overline{\sigma}^2_{yy}$	Diff %	$\overline{\sigma}^3_{yy}$	Diff %	Avg %
	Present-ADT	255.7005	−1.26	60.6788	0.54	47.0467	0.91	9.4093	0.74	38.5267	0.09	30.2048	0.36	6.0410	−1.95	0.84
	Present-EDT	258.3292	−0.25	60.3540	0.00	46.9540	0.71	9.3908	0.54	38.5308	0.10	30.2769	0.60	6.0554	−1.71	0.56
	Present-HDT	255.8456	−1.21	60.6718	0.53	47.0362	0.89	9.4072	0.72	38.5315	0.11	30.2065	0.36	6.0413	−1.94	0.82
	Present-LDT	256.6375	−0.901	60.4035	0.08	47.0271	0.87	9.4054	0.70	38.4625	−0.07	30.2365	0.46	6.0473	−1.85	0.705
	Present-TDT	256.3807	−1.00	60.6424	0.48	46.9970	0.80	9.3994	0.64	38.5475	0.15	30.2128	0.38	6.0426	−1.92	0.77
	Exact [81]	258.97	0.00	60.353	0.00	46.623	0.00	9.34	0.00	38.491	0.00	30.097	0.00	6.161	0.00	0.00
	Touratier [88]‡	253.989	−1.92	60.123	−0.38	47.097	1.02	9.419	0.85	38.249	−0.63	30.187	0.30	6.037	−2.01	1.02
	Karama [88]‡	253.638	−2.06	60.124	−0.38	46.703	0.17	9.34	0.00	38.242	−0.65	30.02	−0.26	6.004	−2.55	0.87
	Levinson [88]‡	253.754	−2.01	59.95	−0.67	46.655	0.07	9.331	−0.10	38.191	−0.78	30.018	−0.26	6.003	−2.56	0.92
	Present-ADT	154.6529	−2.97	65.7448	0.01	49.7774	0.03	4.9777	0.03	43.4120	−0.02	33.5555	−0.01	3.3556	−0.01	0.44
	Present-EDT	155.7830	−2.26	65.3116	−0.64	49.8481	0.17	4.9848	0.17	43.3464	−0.17	33.6771	0.35	3.3677	0.35	0.59
	Present-HDT	154.7853	−2.88	65.7350	0.00	49.7613	0.00	4.9761	0.00	43.4213	0.00	33.5595	0.00	3.3559	0.00	0.41
	Present-LDT	153.7064	−3.560	65.4249	−0.47	50.0350	0.55	5.0035	0.55	43.1832	−0.55	33.5881	0.09	3.3588	0.09	0.836
	Present-TDT	155.2540	−2.59	65.6953	−0.06	49.7055	−0.11	4.9706	−0.11	43.4524	0.07	33.5740	0.04	3.3574	0.04	0.43
	Exact [81]	159.38	0.00	65.332	−0.61	48.857	−1.82	4.903	−1.47	43.566	0.33	33.413	−0.44	3.5	4.29	1.28
	Touratier [88]‡	153.139	−3.92	65.05	−1.04	50.206	0.89	5.02	0.88	43.015	−0.94	33.653	0.28	3.365	0.27	1.17
	Karama [88]‡	153.357	−3.78	65.1	−0.97	49.499	−0.53	4.949	−0.55	43.059	−0.83	33.379	−0.54	3.337	−0.56	1.11
	Levinson [88]‡	152.664	−4.21	65.008	−1.11	49.684	−0.16	4.968	−0.16	42.945	−1.10	33.394	−0.49	3.339	−0.50	1.10
	Present-ADT	115.9661	−4.73	67.2751	0.02	49.7762	0.04	3.3184	0.04	45.9891	−0.03	35.0892	−0.02	2.3393	−0.02	0.70
	Present-EDT	115.6957	−4.95	66.8094	−0.68	50.1411	0.77	3.3427	0.77	45.7550	−0.54	35.2143	0.34	2.3476	0.34	1.20
	Present-HDT	116.0807	−4.63	67.2638	0.00	49.7571	0.00	3.3171	0.00	46.0012	0.00	35.0945	0.00	2.3396	0.00	0.66
	Present-LDT	113.3695	−6.860	66.9882	−0.41	50.4566	1.41	3.3638	1.41	45.4878	−1.12	35.0764	−0.05	2.3384	−0.05	1.614
	Present-TDT	116.4620	−4.32	67.2196	−0.07	49.6972	−0.12	3.3131	−0.12	46.0386	0.08	35.1135	0.05	2.3409	0.05	0.69
	Exact [81]	121.72	0.00	66.787	−0.71	48.299	−2.93	3.232	−2.57	46.424	0.92	34.955	−0.40	2.494	6.60	2.02
	Touratier [88]‡	113.964	−6.37	66.544	−1.07	50.679	1.85	3.378	1.83	45.431	−1.24	35.278	0.52	2.351	0.49	1.91
	Karama [88]‡	114.585	−5.86	66.621	−0.96	49.663	−0.19	3.31	−0.22	45.546	−0.99	34.919	−0.50	2.327	−0.54	1.32
	Levinson [88]‡	113.088	−7.09	66.539	−1.08	50.043	0.57	3.336	0.57	42.293	−8.06	34.903	−0.55	2.326	−0.58	2.64

‡ HSDT-Mesh free method.

Table 8
Symmetric laminated plate under uniaxial load with $a/h = 10$.

n	Theories	Modulus ratio (E_1/E_2)										
		3	Diff %	10	Diff %	20	Diff %	30	Diff %	40	Diff %	Average %
3	Present-ADT	5.3949	1.71	9.8500	0.90	14.9406	−0.52	18.9736	−1.71	22.2686	−2.68	1.50
	Present-EDT	5.3812	1.45	9.7963	0.35	14.7966	−1.48	18.7214	−3.02	21.9015	−4.28	2.12
	Present-HDT	5.3943	1.69	9.8478	0.88	14.9347	−0.56	18.9631	−1.77	22.2534	−2.74	1.53
	Present-LDT	5.3854	1.53	9.8136	0.53	14.8390	−1.20	18.7893	−2.67	21.9924	−3.88	1.96
	Present-TDT	5.3924	1.66	9.8406	0.80	14.9150	−0.69	18.9282	−1.95	22.2021	−2.97	1.61
	Grover et al. [54]	5.3949	1.7	9.8503	0.9	14.9415	−0.5	18.9750	1.7	22.2700	2.7	1.50
	Putcha & Reddy [23]	5.3933	1.7	9.9406	1.8	15.2980	1.9	19.6741	1.9	23.3400	2.0	1.86
	Exact [2]	5.3044	0.0	9.7621	0.0	15.0191	0.0	19.3040	0.0	22.8807	0.0	0.00
5	Present-ADT	5.41600	1.70	10.1380	1.78	15.925	1.74	20.8197	1.73	25.0248	1.76	1.74
	Present-EDT	5.41230	1.63	10.1204	1.61	15.876	1.43	20.7341	1.31	24.9009	1.25	1.45
	Present-HDT	5.41517	1.68	10.1342	1.75	15.914	1.67	20.8010	1.64	24.9976	1.65	1.68
	Present-LDT	5.40225	1.44	10.0716	1.12	15.740	0.55	20.4927	0.13	24.5497	−0.18	0.68
	Present-TDT	5.41230	1.63	10.1204	1.61	15.876	1.43	20.7341	1.31	24.9009	1.25	1.45
	Grover et al. [54]	5.4163	1.7	10.1390	1.8	15.9287	1.8	20.8263	1.8	25.0344	1.8	1.76
	Putcha & Reddy [23]	5.4096	1.6	10.1500	1.9	16.0080	2.3	20.9990	2.6	25.3080	2.9	2.25
	Exact [2]	5.3255	0.0	9.9603	0.0	15.6527	0.0	20.4663	0.0	24.5929	0.0	0.00
9	Present-ADT	5.4201	1.59	10.2099	1.67	16.1896	1.72	21.3386	1.80	25.8258	1.90	1.74
	Present-EDT	5.4169	1.53	10.1978	1.55	16.1577	1.52	21.2815	1.53	25.7403	1.57	1.54
	Present-HDT	5.4193	1.58	10.2072	1.65	16.1826	1.68	21.3260	1.74	25.8070	1.83	1.69
	Present-LDT	5.4078	1.36	10.1610	1.19	16.0583	0.90	21.1026	0.67	25.4717	0.51	0.93
	Present-TDT	5.4169	1.53	10.1978	1.55	16.1577	1.52	21.2815	1.53	25.7403	1.57	1.54
	Grover et al. [54]	5.4202	1.6	10.2100	1.7	16.1911	1.7	21.3413	1.8	25.8298	1.9	1.75
	Putcha & Reddy [23]	5.4313	1.8	10.1970	1.5	16.1720	1.6	21.3150	1.7	25.7900	1.8	1.68
	Exact [2]	5.3352	0.0	10.0417	0.0	15.9153	0.0	20.9614	0.0	25.3436	0.0	0.00

Table 9
Three layered sandwich plate $[0^\circ/C/0^\circ]$ under uniaxial load.

Theories	R					
	1	2	3	4	5	10
Present-ADT	2.7934	3.1843	3.5666	3.9357	4.2915	5.8881
Present-EDT	2.7830	3.1656	3.5376	3.8949	4.2373	5.7515
Present-HDT	2.7930	3.1835	3.5654	3.9341	4.2894	5.8831
Present-LDT	2.7883	3.1731	3.5476	3.9077	4.2531	5.7852
Present-TDT	2.7914	3.1807	3.5612	3.9282	4.2816	5.8635
Exact [89]	2.77	3.33	–	–	4.046	4.2
Pandit et al. [91]	2.754	3.313	–	–	4.029	4.188
Di scuiva [90]	2.843	3.41	–	–	4.136	4.283
Grover et al. [54]	2.7935	3.1844	3.5667	3.9359	4.2916	5.8888

4.3. Free vibration analysis

4.3.1. Symmetric four layered $[0^\circ/90^\circ/90^\circ/0^\circ]$ laminated plate

The effect of modulus rigidity (E_1/E_2) on fundamental frequency $\varpi = \frac{\omega a^2}{h}\sqrt{\rho/E_2}$ is analyzed using the developed theories for four layered symmetric cross-ply $[0^\circ/90^\circ/90^\circ/0^\circ]$ laminated plate with side to thickness ratio 5. The elastic properties EC2 is used and the calculation is carried out by varying the modulus rigidity as 3, 10, 20 and 40. From Table 10 it is noticed that for low modulus rigidity the ADT predicts the fundamental frequency more accurately than those of other shear deformation theories [41,40,92]. However, from the overall observation it is seen that the present EDT yields to be better than the compared results. Further, it should be noticed that HDT and ADT are better than [40,92], however, lesser than [41]. The LDT is well agreement with Grover et al. [92] and better than Ayodogdu [40] and TDT.

4.3.2. Asymmetric five layered sandwich plate

In this final example, a five layered $[0^\circ/90^\circ/C/0^\circ/90^\circ]$ sandwich plate is considered. The elastic properties of core as $E_1 = 2G = 6.89 \times 10^{-3}$ GPa, $rho = 97$ kg/m^3, whereas the elastic properties of face sheets are obtained from EC4. The fundamental frequencies $\varpi = \frac{\omega a^2}{h}\sqrt{\rho/E_2}$ are calculated for side to thickness ratio 10 and 100 as shown in Fig. 5. The thickness ratio of core to the face sheets (t_c/t_f) taken as 10. The evaluated results are compared with [93,94]. Kant and Swaninathan [93] used 12 and 9 number of unknowns for Ref. [93]* and Ref. [93]** respectively as shown in Figs. 5 and 6. For Fig. 6, the analysis is performed by changing the t_c/t_f ratio 4–50 with side to thickness ratio10. From the Figs. 5 and 6 it is clearly observed that HDT provides excellent accuracy for free vibration analysis of sandwich plate those of other shear deformation theories [93]. The ADT and TDT also give augmented results than the compared results [93]. The LDT and EDT are more accurate than Kant and Swaminathan [93] however, behaves worse than above discussed theories.

5. Summary

In this present work an attempt have been made to analyze the influence of various shear strain functions such as algebraic (ADT), exponential (EDT), hyperbolic (HDT), logarithmic (LDT) and trigonometric (TDT) shear deformation theories on static, buckling and free vibration analysis of composite and sandwich plates. The generalized form of governing differential equations

Table 10
Fundamental frequency for four layered [0°/90°/90°/0°] laminated plate with $a/h = 5$.

Refrences	Modulus ratio									Avg %
	3	Diff%	10	Diff%	20	Diff%	30	Diff%	40	
Present-ADT	6.5734	−0.67	8.3085	1.20	9.5883	0.30	10.3501	0.76	10.8767	0.542
Present-EDT	6.5434	−1.13	8.2335	0.29	9.5633	0.03	10.1952	−0.75	10.8422	0.362
Present-HDT	6.5720	−0.69	8.3050	1.16	9.5827	0.24	10.3432	0.69	10.8690	0.522
Present-LDT	6.5508	−1.01	8.2489	0.47	9.4855	−0.78	10.2161	−0.54	10.7182	0.567
Present-TDT	6.5674	−0.76	8.2930	1.01	9.6362	0.80	10.3193	0.46	10.9511	0.643
Exact [37]	6.6180	0.00	8.2100	0.00	9.5600	0.00	10.2720	0.00	10.7520	0.000
Grover et al. [92]	6.6625	0.67	8.2982	1.07	9.6051	0.47	10.2845	0.12	10.7370	0.555
Mantari [41]	6.5650	−0.80	8.2860	0.93	9.5520	−0.08	10.3050	0.32	10.8260	0.453
Aydogdu [40]	6.5460	−1.09	8.2430	0.40	9.4840	−0.79	10.2200	−0.51	10.7280	0.571

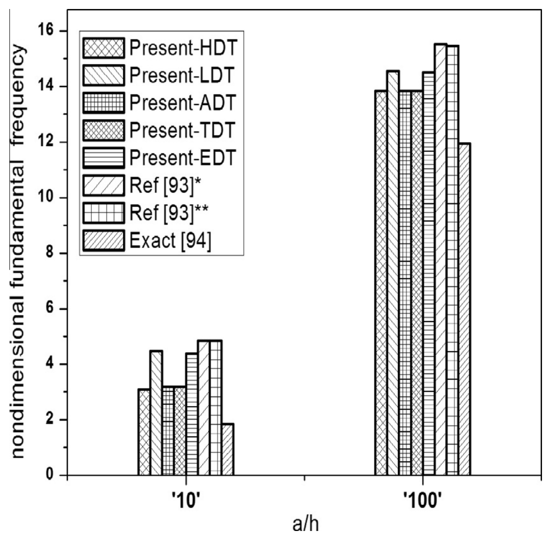

Fig. 5. Effect of a/h on fundamental frequency with $t_c/t_f = 10$ for asymmetric five layered sandwich plate.

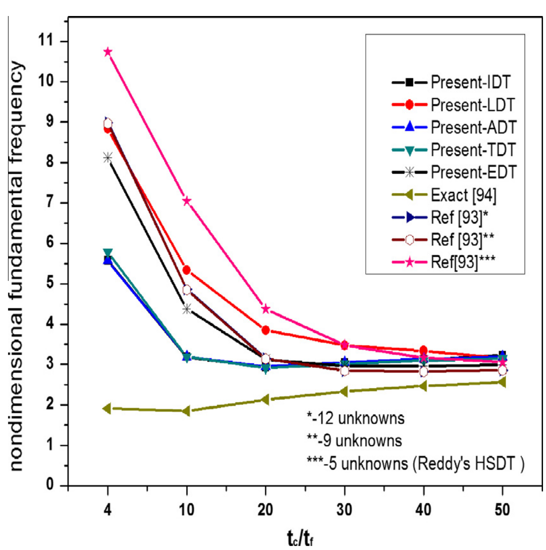

Fig. 6. Effect of t_c/t_f ratio on fundamental frequency for asymmetric five layered sandwich plate.

and associated boundary conditions of the present plate system is achieved through virtual work principle and calculus of variations. To achieve the structural responses, Navier closed type solution technique is employed on the governing differential equations. Though, this work concerned to cross-ply laminated plate with simply supported boundary conditions providing the exact solution without carrying computational and numerical error. A generalized form of governing differential equations are derived involving shear strain function with 5 number of unknowns. The present theories represent nonlinear variations of transverse shear stresses and in-plane displacements. Moreover, the transverse shear stresses completely turned to zero at the plate surfaces. Therefore, an artificial shear correction coefficient is evaded. Overall error percentage with respect to 3D elasticity solution is carried out to justify the accuracy of the proposed models compared to the published theories. From the results discussion, it is clearly noticed that the proposed ADT, HDT, and TDT gives improved results than the existing theories. The EDT and LDT are lesser than the above noted theories nevertheless, gives improved results than the well-known theory.

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Appendix A

Components of $[\overline{K}]$ are

$$\overline{K}_{11} = A_{11}\alpha^2 + A_{66}\beta^2;\ \overline{K}_{12} = (A_{12} + A_{66})\alpha\beta;$$
$$\overline{K}_{13} = -(B_{11}\alpha^2 + B_{12}\beta^2 + 2B_{66}\beta^2)\alpha;$$
$$\overline{K}_{14} = [(\aleph B_{11} + E_{11})\alpha^2 + (\aleph B_{66} + E_{66})\beta^2];$$
$$\overline{K}_{15} = (\aleph B_{12} + E_{12} + \aleph B_{66} + E_{66})\alpha\beta;$$
$$\overline{K}_{22} = A_{66}\alpha^2 + A_{22}\beta^2;\ \overline{K}_{23} = -(B_{22}\beta^2 + B_{12}\alpha^2 + 2B_{66}\alpha^2)\beta;$$
$$\overline{K}_{24} = (\aleph B_{66} + E_{66} + \aleph B_{12} + E_{12})\alpha\beta;$$
$$\overline{K}_{25} = [(\aleph B_{66} + E_{66})\alpha^2 + (\aleph B_{22} + E_{22})\beta^2];$$
$$\overline{K}_{33} = \left[D_{11}\alpha^4 + (2D_{12} + 4D_{66})\alpha^2\beta^2 + D_{22}\beta^4\right];$$
$$\overline{K}_{34} = -\alpha\left[(\aleph D_{11} + F_{11})\alpha^2 + (\aleph D_{12} + F_{12})\beta^2 + 2(\aleph D_{66} + F_{66})\beta^2\right];$$
$$\overline{K}_{35} = -\beta\left[(\aleph D_{12} + F_{12})\alpha^2 + (\aleph D_{22} + F_{22})\beta^2 + 2(\aleph D_{66} + F_{66})\alpha^2\right];$$
$$\overline{K}_{44} = \left[\aleph(\aleph D_{11} + F_{11})\alpha^2 + (\aleph F_{11} + H_{11})\alpha^2 + \aleph(\aleph D_{66} + F_{66})\beta^2 + (\aleph F_{66} + H_{66})\beta^2 + \aleph^2 A_{55} + 2\aleph R_{55} + S_{55}\right];$$
$$\overline{K}_{45} = [\aleph(\aleph D_{12} + F_{12}) + (\aleph F_{12} + H_{12}) + \aleph(\aleph D_{66} + F_{66}) + (\aleph F_{66} + H_{66})]\alpha\beta;$$
$$\overline{K}_{55} = \left[\aleph(\aleph D_{22} + F_{22})\beta^2 + (\aleph F_{22} + H_{22})\beta^2 + \aleph(\aleph D_{66} + F_{66})\alpha^2 + (\aleph F_{66} + H_{66})\alpha^2 + \aleph^2 A_{44} + 2\aleph R_{44} + S_{44}\right]$$

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